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AMTI

THE MATHEMATICS TEACHER

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disserminate new trands in mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equpping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmers, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds "National Mathematical Talent Search Competition" annually and organises Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

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"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

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The Mathematics Teacher
(INDIA)

OFFICIAL JOURNAL OF THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

Volume: 58 Issues: 1&2 Year: 2022

Meghraj Bhatt
Editor

EDITORIAL

This is volume 58 (1&2) of this journal of the The AMTI. It contains question papers and solutions of the NMTC-53. As usual4 question papers of screening test,4 question papers of Final Test and one for Senior Level are included. I hope this will be useful to the teachers to guide the students to prepare for such talent test. Also students can use it as a practice book so that they can understand the nature of the questions.

- **Meghraj Bhatt**

Note : This volume 58 (1 & 2) pertaining to the year 2022 – 23 is being published now and the delay in publishing is regretted.

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NMTC-53 – PRIMARY – MARCH – 2022
SCREEN TEST - GAUSS CONTEST
(V & VI STANDARDS)

1. If (8*4)+6–(10÷5)=8,then * stands for the operation
   a) + b) × c) ÷ d) –

   Solution:

   $(8*4)+6-(10\div5)=8$
   $\therefore (8*4)+6-2=8$
   $\therefore (8*4)+4=8$
   $\therefore 8*4=4$
   But $8-4=4$ $\therefore$ * stands for – . Ans:d

2. 20% of a number is equal to 30% of another number. Six times the bigger of these numbers added to the smaller number is 2000.Then 10% of the smaller numer is —
   a) 10 b) 20 c)30 d) 25

   Solution:

   Let x be the bigger and y be the smaller number.

   Given
   $\frac{20}{100}xx=\frac{30}{100}xy \quad \Rightarrow 2x=3y \; ----(i)$
   Also given $6x+y = 2000 \; ----- \; (ii)$

   Using (i) in (ii) $9y + y = 2000 \quad \Rightarrow y = 200$

   $\therefore$ 10% of 200 =20 . Ans: b

3. Samrud chooses a two digit number. He subtracts it from 200 and doubles the result. The largest number he can get is
   a) 390 b)380 c)400 d) 370

Solution:
If samrud subtracts the smallest two digit number 10 from 200,he gets 190. When he doubles it, he get 380. Ans :b

Alternate method:
Let the two digit number be 10a+b. subtracting if from 200, we get 200-(10a+b).Doubling the result, we get 400-2(10a+b). If this is to be largest, we must have 2(10a+b) to be smallest i.e. 10 a+b to be smallest i.e.10.
∴The largest number obtained is 380. Ans :b

4. The least number which leaves 2 as remainder when divided by 5,6,8,9, and 12 is
a) 362 b)342 c)332 d) 322

Solution:
LCM of 5,6,8,9,12 is 360.To get a remainder 2, we have 362.
Ans :a

5. Saket takes the number 2021. There are two 2s, one 0 and one 1.Using these numbers,he makes 3-digit numbers. The sum of all these number is
a) 1488 b)150 c)1409 d) 1610

Solution:
Starting with 1, the three digit numbers are 122,102,120.
Starting with 20, they are 201,202.
Starting with 21, they are 210,212
Starting with 22, they are 220,221
∴Total = 1610. Ans: d

6. In the net of the adjoining figure,numbers 1 to 6 are marked. The net is folded to form a cube. The number that appears on the opposite face of 6 is
a) 4 b) 3 c)2 d) 1

Solution:
If we mentally fold it to form a cube, we see that, opposite to 6 will be 2. Ans:c

7. $10^{2021} - 1$ is written as an integer in the usual decimal form. The sum of all digits of this written number is
a) 18189 b)18187 c)18199 d) 19189

Solution:
$10^1 - 1 = 9, \quad 10^2 - 1 = 99, \quad 10^3 - 1 = 999 \ldots\ldots$

$$\therefore 10^{2021} - 1 = \underbrace{999.....9}_{2021}$$

$$\therefore \text{sum of all digits} = \underbrace{9+9+9+.....+9}_{2021} = 9 \times 2021 = 18189.$$

Ans :a

8. August 15th 1992 was a Saturday. What day was 15th August,1991?
a) Monday b)Wednesday c)Thursday d) Sunday.

Solution:
In the given interval of 1 year, the month of February falls in 1992 which was a leap year. Thus this 1 year interval would be of 366 days which gives 2 add days.
$\therefore$ August 15,1991 would be two days before Saturday.
$\therefore$ It was a Thursday. Ans:c

9. Two thirds of the students in a class room are seated in three fourths of the chairs. The rest of the students are punished for not submitting the homework and hence asked to stand. There are 6 empty chairs.The numbers of persons (that is the number of students including the teacher)in the classroom is
a) 27 b) 28 c) 31 d) 32

Solution:

Let n be the number of students and c be the number of chairs.

By given date, $\frac{1}{4}$ of the total chain are not used and empty chain are 6 $\therefore \frac{c}{4} = 6 \Rightarrow c = 24.$

Again $\frac{2}{3}n = \frac{3}{4}c \Rightarrow \frac{2}{3}n = 18 \Rightarrow n = 27.$

Including the teacher, no.of persons=28. Ans :b

10. There are 24 four-digit numbers with different digits, formed by 2,4,5 and 7. One of these 4-digit numbers is a multiple of another. Which one of the following is it?

a) 7245 b)7542 c)7254 d) 7425

Solution:

6 numbers starts with 2
6 numbers starts with 4
6 numbers starts with 5
6 numbers starts with 7

Doubling and tripling the numbers in 2000's may produce a solution but any multiple of the others is larger than 8000.

Unit digits of the numbers are 2,4,5,7.Their doubles are 4,8,0,4. Consider choice (c).7254 produces $\frac{7254}{2} = 3627$ which is not in the list. When unit digits are tripled we get 6,2,5,1.So we have to check the numbers ending with 2 or 5.

7542÷3=2514 which is not.
7245÷3=2415 which is not
7425÷3=2475 which is not

The correct number. Ans:d

Fill in the blanks:

11. Twelve friends went to a restaurant to have lunch. They ordered for 12 meals. When they were served, they found that the food was too much which can be shared by 18 people. The number of meals they would order to cater to only 12 is _.

Solution:

12 meals caters to 18 people.

$\therefore$ 18 people receives 12 meals.

$\therefore$ 1 person receives $\frac{12}{18}=\frac{2}{3}$ meals.

$\therefore$ The number of meals sufficient for 12 people $=\frac{2}{3}\times 12=8$.

Ans. 8

12. The least multiple of 23 which when divided by 18, 21 and 24 leaves the remainders 7, 10, and 13 respectively is _____.

Solution:

Find LCM of 18, 21, and 24.

2	18, 21, 24
3	9, 21 12
	3, 7, 4

$\therefore$ LCM $= 2 \times 3 \times 3 \times 7 \times 4 = 504$.

We observe that in all cases, the remainder is just 11 less than the divisor.

$\therefore$The number will be of the form 504k-11. Where k is a positive integer.

Now $\therefore 23x21 = 483$

$\therefore 504k-11 = (483+21)k-11$

$= 483k + 21k-11.$

483k is a multiple of 23.

(504k-11) is a multiple of 23, then (21k-11) must be divisible by 23,k=1,2,3-----We find the minimum value of k for which 21k-11 is divisible by 23 is 6, because 21x6-11=126-11=115=23x5.

$\therefore$ The required number is 504x6-11=3013.

Ans. 3013

13. The value of $\sqrt{\frac{(0.1)^2+(0.01)^2+(0.008)^2}{(0.01)^2+(0.001)^2+(0.0008)^2}}$ is

Solution:

Given expression

$$= \sqrt{\frac{\left(\frac{1}{10}\right)^2+\left(\frac{1}{100}\right)^2+\left(\frac{8}{1000}\right)^2}{\left(\frac{1}{100}\right)^2+\left(\frac{1}{1000}\right)^2+\left(\frac{8}{10000}\right)^2}} = \sqrt{\frac{\left[\left(\frac{1}{10}\right)^2+\left(\frac{1}{100}\right)^2+\left(\frac{8}{1000}\right)^2\right] x \frac{100}{100}}{\left(\frac{1}{100}\right)^2+\left(\frac{1}{1000}\right)^2+\left(\frac{8}{10000}\right)^2}}$$

$$= \sqrt{\frac{\left[\left(\frac{1}{100}\right)^2+\left(\frac{1}{1000}\right)^2+\left(\frac{1}{10000}\right)^2\right]100}{\left(\frac{1}{100}\right)^2+\left(\frac{1}{1000}\right)^2+\left(\frac{1}{10000}\right)^2}} = \sqrt{100} = 10$$

Ans. 10

14. Two numbers are respectively 20% and 50% of a third number. The percentage of the first number to the second is -------

Solution:

Let the third number be n.

First number $= \frac{20}{100} xn = \frac{n}{5}$

Second number $= \frac{50}{100} xn = \frac{n}{2}$

Required percentage $= \frac{n/5}{n/2} x100 = 40$

$\therefore$ The first number is 40% of the second.

Ans. 40 %

15. Water flows through a rectangular opening 3mx2m with a speed of 1.5km/hr. The amount of water that passes through the opening in 5 minutes is xm^3 .Then the value of x is -----

Solution:

speed of stream $= 1.5\, {}^{km}/_{hr}$

$$= \frac{1.5 * 1000}{60} {}^{mtr}/_{minute}$$

$$= 25\, {}^{mt}/_{minute}.$$

Volume of the water flown = (cross section area of the opening)*(speed of stream of water)*time.

$$= (3 * 2) * (25) * (5)m^3$$

$\therefore vol. = 750m^3$

Ans: $750m^3$

16. *If* $A = \frac{1111+3333+5555}{222+333+444}$, $B = \frac{555+222}{1111+2222+4444}$, then A*B=----

Solution:

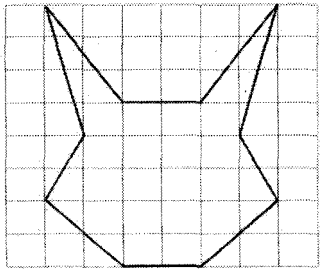
$$A = \frac{1111(1+3+5)}{111(2+3+4)} = \frac{1111*9}{111*9} = \frac{1111}{111}$$

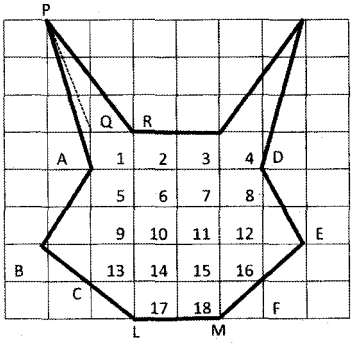
$$B = \frac{111(5+2)}{1111(1+2+4)} = \frac{111*7}{1111*7} = \frac{111}{1111}$$

$$\therefore A * B = \frac{1111}{111} * \frac{111}{1111} = 1.$$

Ans:1

17. In the adjoining figure of a cat, each small square is $1cm^2$ .The area of the cat (in cm^2) is --------





Solution:

(i) square in the face = 18

(ii) $\Delta ABC \cong \Delta DEF$

$with\ base = 3, height = 1$

$\therefore Area = 2\left[\frac{1}{2}.3.1\right] = 3$

(iii)$2(\Delta APQ + \Delta PQR)$

$$= 2\left[\frac{1}{2}.1.1 + \frac{1}{2}.1.3\right] = 4$$

(iv)Two Δs with hypotenuse CL and MF with area = 1

$\therefore$ Total Area =18+3+4+1=26. Ans:26

18. The sum of the digits of the number given by the product

$\underbrace{666....66}_{2021 digits} \times \underbrace{999.....99}_{2021 digits} is ----$

Solution:

$\underbrace{666....66}_{2021 digits} \times (\underbrace{1000.....00}_{2021 zeros} - 1)$

$= \underbrace{666.....66}_{2021 digits}\underbrace{000.....00}_{2021 zeros} - \underbrace{666.....66}_{2021 digits}$

=666.....65333....334

To find the sum of digits,we observe that there are 2020 6's and 2020 3's. i.e,2020*9=18180 and 5+4=9 $\therefore$ Total = 18189

Ans:<u>18189</u>

19. Two buses X,Y started from two different places P_1 and P_2 respectively,moving towards each other. The ratio of the speed of the bus X to that of bus Y is 5:4.After they meet, the speed of the bus X was reduced by 20% and the speed of Y was increased by 20% .When bus X arrived at P_2, bus Y was still 10 kms away from P_1.The distance between places P_1 and P_2 (in kilometres) is ---------

Solution:

After they meet, the speed of bus X and that of Y are in the ratio

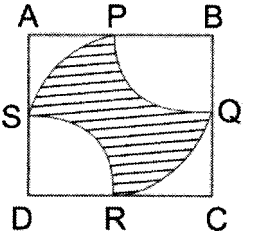
$$5 * (1 - 20\%) : 4(1 + 20\%) = 5:6$$

When they meet, bus x and bus y had completed 5/9 and 4/9 of the whole journey. After they met, x covered 4/9 of the journey to reach P_2 while bus y covered another $\frac{4}{9} * \frac{5}{6}$ of the whole journey.

Distance between the places $= 10 \div (\frac{5}{9} - \frac{4}{9}.\frac{6}{5})$

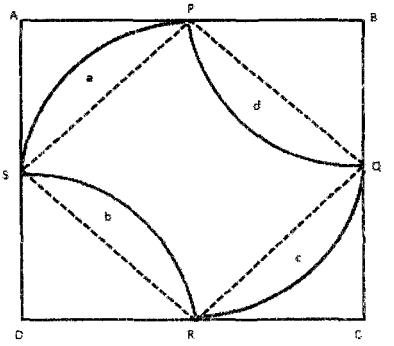
$= 450$ km.

20. In the adjoining diagram, ABCD is a square. P,Q,R,S are the mid points of the sides AB,BC,CD and DA respectively. PQ and SR are quadrants of circles with B,D as centres and BP as radius. PS and QR are part of the circles passing through P,Q,R and S. If AB=20 cm, the area of the shaded regions (in cm^2) is----------



Solution:

The small portions a,b,c,d are of equal area by symmetry. PQRS is a square. a and c are added to the area of the square and b and d are subtracted.



$$\therefore shaded\ area = Area\ of\ the\ square\ PQRS$$
$$= (PQ)^2 = (\sqrt{10^2 + 10^2})^2$$
$$= 200cm^2$$

Ans: $200cm^2$

NMTC-53 – SUB-JUNIOR – MARCH – 2022
SCREENING TEST - KAPREKAR CONTEST
(VII & VIII STANDARDS)

1) If $x + y = 6$, the value of $(x^3 + y^3 + 18xy)$ is

a) 94 b) 100 c) 216 d) 200

Solution:

$$x + y = 6 \Rightarrow (x + y)^3 = 216$$
$$\Rightarrow x^3 + y^3 + 3xy(x + y) = 216$$
$$\Rightarrow x^3 + y^3 + 18xy = 216$$

Ans: c

2) The radius and vertical height of a cone are respectively r and $2h$. The radius and height of a cylinder are respectively $2r$ and h. If V_1 and V_2 are respectively the volumes of the cone and the cylinder, then $V_1 : V_2$ is

a) 1 : 3 b) 1 : 4 c) 1 : 6 d) 1: 5

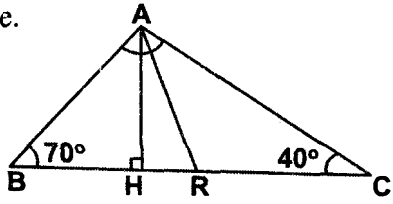
Solution:

$$\frac{V_1}{V_2} = \frac{\frac{1}{3}\pi . r^2 (2h)}{\pi (2r)^2 . h} = \frac{1}{6}$$

Ans:C

3) In the adjoining figure AR is the bisector of $\angle BAC$ and AH is the altitude. The measure of $\angle HAR$ is

a) 20° b) 15°
c) 28° d) 10°



Solution:

Let $\angle HAR = \theta . ° \angle BAH = 20°$

$\therefore \angle BAR = 20° + \theta° - - - - - - - - - - - - - - - (1)$

In $\Delta ABC, \angle BAC = 180° - (70° + 40°) = 70°$

$\therefore \angle BAR = 35° - - - - - - - - - - - - - - - - - (2)$

From (1) & (2) $20° + \theta° = 35° \Rightarrow \theta = 15°$ Ans:b

4) The value of x satisfying the equation $\frac{3x-4}{2x+1} = \frac{6x-3}{4x+7}$ is

a) a multiple of 3 b) a multiple of 7

c) a multiple of 4 d) a prime number

Solution:

$$\frac{3x-4}{2x+1} = \frac{6x-3}{4x+7}$$

$$\therefore (3x-4)(4x+7) = (6x-3)(2x+1)$$

On simplification we get

$5x = 25 \Rightarrow x = 5$ Ans:d

5) Given $n = 2021$, the value of $\frac{7^n + 7^{n+1}}{7^{n+2} - 7^n}$ is

a) $\frac{1}{6}$ b) $\frac{1}{8}$ c) $\frac{2}{9}$ d) $\frac{1}{7}$

Solution:

$$\frac{7^n + 7^{n+1}}{7^{n+2} - 7^n} = \frac{7^n(1+7)}{7^n(7^2-1)} = \frac{8}{48} = \frac{1}{6}$$

Note: Here the value of the expression is $\frac{1}{6}$ irrespective of the value of n.

Ans:a

6) If $a+b+c=0$, the numerical value of

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}$$ is

a) 0 b) 1 c) 2 d) $\frac{1}{2}$

Solution:

$$2a^2+bc=a^2+a^2+bc=a^2+a.a+bc$$
$$=a^2-a(b+c)+bc \quad (\text{given } a+b+c=0)$$
$$=a^2-ab-ac+bc$$
$$=a(a-b)-c(a-b)$$
$$=(a-b)(a-c)$$

Similarly $2b^2+ca=(b-c)(b-a)$ and

$2c^2+ab=(c-a)(c-b)$

Given expression

$$=\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$$

$$=\frac{-\left[a^2(b-c)+b^2(c-a)+c^2(a-b)\right]}{(a-b)(b-c)(c-a)}$$

$$=\frac{(a-b)(b-c)(c-a)}{(a-b)(b-c)(c-a)}$$ (Re-arranging the numerator)

=1

Ans:b

7) A cuboid has length 'a' units, breadth 'b' units and height 'c' units. S and V are the total surface area and volume of the cuboid respectively.

Then the numerical value of $\frac{V}{S}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ is

a) 1 b) 2 c) 3 d) $\frac{1}{2}$

Solution:

$v = abc$ and $s = 2(ab+bc+ca)$

$$\therefore \frac{v}{s}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \frac{abc}{2(ab+bc+ca)}\left[\frac{bc+ac+ab}{abc}\right]$$

$$= \frac{1}{2}$$

Ans:d

8) Samrudh divided $\frac{7}{9}$ by $\frac{2}{3}$ and got the answer x; Saket divided $\frac{5}{11}$ by $\frac{15}{22}$ and got the answer y; Vishwa divided x by y; the answer he got is

a) $\frac{7}{4}$ b) $\frac{6}{7}$ c) $\frac{2}{9}$ d) $\frac{1}{9}$

Solution:

samrudh's answer $x = \frac{7}{9}\times\frac{3}{2} = 7/6$

Saket's answer $y = \frac{5}{11}\times\frac{22}{15} = 2/3$

Vishwa's answer $= \frac{7}{6} \times \frac{3}{2} = 7/4$

Ans:a

9) The simplified value of $\sqrt{\frac{(2022)^3 - (2020)^3 - 2}{6}}$ is

a) 2021 b) 1 c) 3 d) $(2021)^2$

Solution:

Let a=2021

Given expression $= \sqrt{\frac{(a+1)^3 - (a-1)^3 - 2}{6}}$

$$= \sqrt{\frac{a^3 + 1 + 3a(a+1) - [a^3 - 1 - 3a(a-1)] - 2}{6}}$$

$$= \sqrt{\frac{a^3 + 1 + 3a^2 + 3a - a^3 + 1 + 3a^2 - 3a - 2}{6}}$$

$$= \sqrt{\frac{6a^2}{6}}$$

=a=2021 Ans:a

10) If $x^{x^5} = 5$, then the value of x is

a) $5^{1/25}$ b) 5^5 c) $\left(\frac{1}{5}\right)^5$ d) $5^{1/5}$

Solution:

$x^{x^5} = 5$

Let $x^5 = a$

$\therefore x^a = 5$ from $x^5 = a$, we get $a^{1/5} = x$

$\therefore \left(a^{1/5}\right)^a = x^a = 5$

$\therefore \left(a^{a/5}\right)^5 = 5^5$

$\therefore a^a = 5^5 \Rightarrow a = 5$

$\therefore x^5 = 5 \qquad \therefore x = 5^{1/5}$

Ans:d

11) If $a \neq b \neq c$, $a^2 + a = b^2, b^2 + b = c^2$, $c^2 + c = a^2$ then the numerical value of $(a-b)(b-c)(c-a)$ is

a) 1 b) 0 c) 2 d) abc

Solution:

$a^2 + a = b^2$, $b^2 + b = c^2$, $c^2 + c = a^2$

Adding all, $a^2 + b^2 + c^2 + a + b + c = a^2 + b^2 + c^2$

$\Rightarrow a + b + c = 0 - - - - - - - (1)$

Now $a^2 + a = b^2 \Rightarrow a^2 - b^2 = -a$

$\Rightarrow (a-b)(a+b) = -a$

But $a + b = -c$ from (i)

$\therefore a - b = \frac{a}{c}$

Similarly $b - c = \frac{b}{a}$ and $c - a = \frac{c}{b}$

Multiplying all, $(a-b)(b-c)(c-a) = \frac{a}{c}.\frac{b}{a}.\frac{c}{b} = 1$

Ans:a

<u>One can think like this also:</u>

$a^2 + a = b^2 \Rightarrow a(a+1) = b^2$ .Put this value of b^2 in second equation,

$a(a+1)+b=c^2$.Put this in third equation

$a(a+1)+b+c=a^2 \Rightarrow a+b+c=0$--------(i)

Again $a^2+a=b^2 \Rightarrow a^2-b^2=-a$ similarly

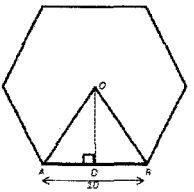
$$b^2-c^2=-b \,\&\, c^2-a^2=-c$$

Multiplying all $(a^2-b^2)(b^2-c^2)(c^2-a^2)=-abc$

$$\Rightarrow (a-b)(b-c)(c-a)=\frac{-abc}{(a+b)(b+c)(c+a)}$$

$$=\frac{-abc}{(-c)(-a)(-b)}=1$$ Ans:a

12) The radius of the circle inscribed in a regular hexagon of side 10 units is r ; then



a) $6<r<7$ b) $7<r<8$ c) $8<r<9$ d) $9<r<10$

clearly, AOB is an equilateral triangle.

$\therefore OA=10cm$

Let $OD=r$ be the radius of the circle inscribed in the hexagon and

$OD \perp AB$ $\quad \therefore AD=5cm$

$\therefore OD=r=\sqrt{10^2-5^2}=5\sqrt{3}\approx 5(1.732)=8.660$

$\therefore 8\angle r\angle 9$ Ans:c

13) Each digit of a large number P is 1. If the number of digits of P is 2021, the value of the sum of the digits of the product $P \times$ 2021 is

a) 2021 b) 2021^2 c) 202100 d) 10105

Solution:

$$p = \underbrace{1111.....1}_{2021 digits}$$

digit sum

$2021 \times 1 = 2021$ $\quad 5 = 5 \times 1$

$2021 \times 11 = 22231$ $\quad 10 = 5 \times 2$

$2021 \times 111 = 224331$ $\quad 15 = 5 \times 3$

.....

.....

$\therefore$ Digit sum of

$2021 \times \underbrace{111.....1}_{2021 digits}$ Will be equal to 5×2021

$= 10105$

Ans:d

14) A retail merchant goes to a wholesale supplier of goods with a certain sum of money, with which he can buy either 100 pens or 80 pencils. Retaining with himself 20% of money for conveyance to go back home, he buys 40 pencils and of the balance he purchases pens. The number of pens he bought is

a) 40 b) 50 c) 30 d) 35

Solution:

Let the price of one pen be x(Rs.)

The person has 100x Rs.

80 pencils cost 100x $\therefore$ 1 pencil cost 1.25x

40 pencils bought for $40 \times 1.25x = 50x$

20% of the total amount = 20xRs.

Money left = $100x - (\text{conveyance}) - (\text{cost of pencils})$

$= 100x - 20x - 50x$

$= 30x$

$\therefore$ He bought 30 pens Ans:c

Alternatively:

Let the total money he had = x Rs.

$\therefore$ cost of 1 pen $= \frac{x}{100}$ Rs. And

Cost of 1 pencil $= \frac{x}{80}$ Rs.

Money for conveyance $= \frac{20x}{100} = \frac{x}{5}$

$\therefore$ Money left $= x - \frac{x}{5} = \frac{4x}{5}$

Spent money on 40 pencils $= 40 \times \frac{x}{80} = \frac{x}{2}$

$\therefore$ money Left $= \frac{4x}{5} - \frac{x}{2} = \frac{3x}{10}$

$\therefore$ No. of pens purchased $= \frac{3x/10}{x/100} = 30$ Ans:c

15) If $\frac{2}{n} + \frac{1}{n^2}$ (where n is a natural number) can be written as an irreducible fraction whose numerator and denominator has no common factor and the sum of the numerator and denominator is 2025; then the numerator is

a) 63 b) 64 c) 91 d) 89

Solution:

$$\frac{2}{n} + \frac{1}{n^2} = \frac{2n+1}{n^2}$$

Given $n^2 + 2n + 1 = 2025 \Rightarrow (n+1)^2 = 2025 \Rightarrow n+1 = 45$

$\Rightarrow n = 44$

$\therefore$ Numerater $= 2n+1 = 2(44)+1 = 89$ Ans:d

Fill in the blanks:

(Note: While answering these questions, do not use commas in between numerals. For example, if you want to give 13489 as answer, do *not* write 13,489; simply write 13489)

16) There are baskets of 500 mangoes sold by 4 roadside vendors. Four persons *A*, *B*, *C* and *D* bought mangoes, each from a different vendor. A got 10% less than the number of mangoes bought by *B*; *B* got 25%more than the number of mangoes got by *C* while *C* got 20% less than *D*'s number of mangoes. If *A* got 360 mangoes, then *D* got x% of mangoes in a basket. Then the value of x is _____.

Solution:

Let D get $x\%$ of the mangoes in the basket.

$$\text{Number of mangoes D gets} = x\%\text{of } 500 = 5x$$

$$\text{Number of mangoes C gets} = 5x\left(\frac{100-20}{100}\right) = 4x$$

$$\text{Number of mangoes B gets} = 4x\left(\frac{100+25}{100}\right) = 5x$$

$$\text{Number of mangoes A gets} = 5x\left(\frac{100-10}{100}\right) = 4.5x$$

Given 4.5x=360 , x=80

Ans:80

17) The ratio of an interior angle to the exterior angle of a regular polygon is 5 : 1. The number of sides of the polygon is ____.

Solution:

$$\text{Each angle of a regular polygon of n sides} = \frac{(2n-4)90^\circ}{n}$$

Sum of all exterior angles of an n-gon=360°

$\therefore$ Each exterior angle $= \frac{360°}{n}$

Given $\frac{\frac{(2n-4)90°}{n}}{\frac{360°}{n}} = 5$

$\Rightarrow \frac{n-2}{2} = 5 \Rightarrow n-2 = 10 \Rightarrow n = 12$

Ans:12

18) Some cows and some hens, totally 20 in all, were seen in a place. When all the legs are counted we get 64. The number of cows is _____.

Solution:

Let therebe 'c'cows and 'h' hens

By the problem c+h=20 and
4c+2h=64

Solving ,
we get number of cows=12 Ans:12

19) Given a, b, c, d, e are natural numbers such that $a + b = b + c = c + d = d + e = 4021$ and $a + b + c + d + e = 11063$. Then the numerical value of $(a — d)$ is _____.

Solution:

$a+b=b+c \Rightarrow a=c$

$b+c=c+d \Rightarrow b=d$

$c+d=d+e \Rightarrow c=e$

Now $a+b+c+d+e=11063$

$\Rightarrow a+b+a+b+a=11063 \quad \Rightarrow 3a+2b=11063$

$\Rightarrow 2(a+b)+a=11063$

$\Rightarrow 2(4021)+a=11063$ (given)

$\Rightarrow a=3021$

Again $a+b=4021$ and $a=3021 \Rightarrow b=1000=d$

And $a-d=3021-1000=2021$ Ans:2021

20) The sum of the digits of a two-digit number is 6. If we add 18 to this number, we get a two-digit number consisting of the same digits in the reverse order. The sum of the squares of the digits of the original two-digit number is _____.

Solution:

Let $(10x+y)$ be the two-digit number.

Given $x+y=6$ ----------(1)

$10x+y+18=10y+x \Rightarrow x-y=-2$ ----------(2)

Solving (1)&(2) we get $x=2y=4$

$\therefore x^2+y^2=4+16=20$ Ans:20

21) The four natural numbers n, m, $n—m$, $n+m$ (where $n > m$) are all primes. The least numerical value of the sum of all these primes is _____.

Solution:

$n > m$; n, m are primes.

$\therefore n \neq 2$ If $n=3$, then,

$m=2, \therefore n-m=1$ which is not a prime.

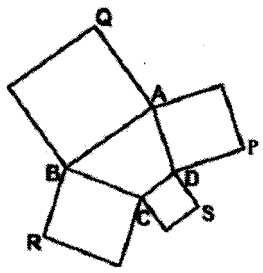
If $n=5, m=2 or 3$. Then

$n=5, m=2 \Rightarrow n-m=3, n+m=7$ are primes.

$n=5, m=3 \Rightarrow n-m=2, n+m=8$ not a prime

$\therefore$ The primes are 5,2,3,7 and their sum =17 Ans:17

22) In the adjoining figure, ABCD is a quadrilateral. Squares are drawn on each side as in the figure. AP = $4\sqrt{2}$ units. Area of ΔABQ = 18 (units)^2. Area of square on BC is 25 (units)^2 and the perimeter of the square on CD is 18 units. The perimeter of the quadrilateral



ABCD (in the same unit) is $\frac{p}{q}$ where p , q are natural numbers which have no common factors. Then the value of $(p + q)$ is _____.

Solution:

$AD = 4$ since diagonal $AP = 4\sqrt{2}$

$$\text{Area of } \Delta ABQ = 18(unit)^2$$

$$\therefore \text{Area of sq.m } AB = 36(unit)^2$$

$$\therefore AB = 6$$

Area of the square on $BC = 25$ $\therefore BC = 5$

Perimeter of the square on $CD = 18$ $\therefore CD = 4.5$

Now peri =4+6+5+4.5=19.5=$\frac{39}{2}$ units

$\therefore p = 39, q = 2$ $\therefore p + q = 41$

Ans:41

23) My friend Mr. Jack, has reached a square age. Mrs. Jack's age is the product of the digits of her husband's age. They have a son and a daughter. The age of the son is the sum of the digits of the age of the mother while the age of the daughter is the sum of the digits of the father. The age of the son is _____.

Solution:

Let us make a table of all the possibilities.

Age(in years)	Possibilities			
Mr.Jack	25	36	49	64
Mrs.Jack	10	18	36	24
Daugher	7	9	13	10
Son	1	9	9	6

Obviously, the age of the son is 9 years.

Note: considering older ages of mr.Jack will not give feasible figures.

24) The lengths of the sides of a triangle (in centimetres) are three

consecutive integers. The length of the shortest side is 25% of its perimeter. The area of the triangle (in square centimetres) is _____.

Solution:

Let the sides be $n-1, n, n+1$ centimeters.

$\therefore Perimeter = 3n$

given shortest side is 25% of the perimeter,

$$n-1 = \frac{3n}{4} \Rightarrow n = 4$$

$\therefore$ The sides are 3,4,5 cms and it is a right triangle.

$\therefore$ The area $= \frac{1}{2}.3.4 = 6 sq.cm$ Ans:6

25) Amar lists some four positive integers in some order, such that 1 more than the first, 2 less than the second, 3 times the third and one-fourth of the last number are all the same. If the sum of the four numbers is 58, the largest number in the list is _____.

Solution:

Let a,b,c,d be the numbers listed.

By given data, $a+1 = b-2 = 3c = \frac{d}{4} = k(say)$ -------(1)

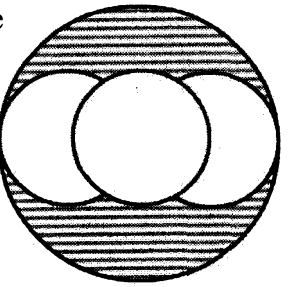
$a+b+c+d = 58$. Using (1) we get,

$$(k-1)+(k+2)+\frac{k}{3}+4k = 58 \Rightarrow k = 9$$

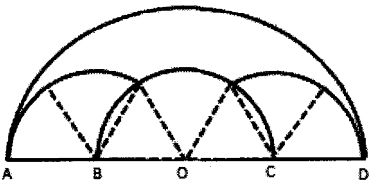
$\therefore$ The numbers are 8,11,3 and 36.

$\therefore$ Largest is 36. Ans:36

26) In the adjoining figure, the radius of the largest circle is 2 *cm* and the radius of each of the three equal circles is 1 *cm*. The middlemost circle is concentric with the largest one. The area of the shaded portion is $\frac{7\pi - 3\sqrt{a}}{3}$ square units. Then the value of 'a' is _____.



Solution:
Consider half of the figure.



Area of the largest semicircle $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi.4 = 2\pi - -(1)$

$AB = BO = OC = CD = 1cm$. All dotted segment are of 1c.m. and all angles at B,O and C are of $60°$.

Shaded Area$=2\pi - 5\,(area\ of\ sectors) - 2(area\ of\ triangles)$

$$= 2\pi - 5\frac{\pi}{6} - 2\frac{\sqrt{3}}{4} = \frac{7\pi - 3\sqrt{3}}{6}$$

$\therefore$ Total shaded area $= \frac{7\pi - 3\sqrt{3}}{3}$

$\therefore a = 3$ Ans:3

27) There are four two-digit numbers. None of the digits is zero; no two of them are the same. In all the 8 digits used, one digit is not included. If their sum is 221, the excluded digit is ____.

Solution 1:

If $a_1b_1, a_2b_2, a_3b_3, a_4b_4$ are the two-digit numbers then

$b_1 + b_2 + b_3 + b_4 = 31,21,11$

$a_1 + a_2 + a_3 + a_4 = 19,20,21$

$a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ all are different and

$1+2+3+4+5+6+7+8+9 = 45$

$\therefore$ The second possibility only can happen and 4 is not included in the digits.

Ans:4

Solution:2

$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = 221$

The digit sum is

$a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4 = 9m + 5$

$\therefore$ The missing digit is 4.

Ans:4

28) If a, b, c are natural numbers such that $a + \dfrac{1}{b + \dfrac{1}{c}} = \dfrac{37}{16}$, then the numerical value of $a+b+c$ is _____.

Solution:

$$a + \frac{1}{b + \frac{1}{c}} = \frac{37}{16} = 2 + \frac{5}{16} \qquad \therefore a = 2$$

Now, $\dfrac{1}{b + \dfrac{1}{c}} = \dfrac{5}{16}$

$$\therefore b + \frac{1}{c} = \frac{16}{5} = 3 + \frac{1}{5}$$

$\therefore b = 3, c = 5 \qquad \therefore a+b+c = 10$ Ans.

Alternatively:

$$a+\frac{1}{b+\frac{1}{c}}=\frac{37}{16}=a+\frac{1}{b+\frac{1}{c}}=2+\frac{5}{16}$$

$$\Rightarrow a+\frac{1}{b+\frac{1}{c}}=2+\frac{1}{\frac{16}{5}}\Rightarrow a+\frac{1}{b+\frac{1}{c}}=2+\frac{1}{3+\frac{1}{5}}$$

$\therefore a=2, b=3, c=5$

$\therefore a+b+c=10$ Ans:10

29) Given a, b, c are *real* numbers such that $a-b=8$ and $ab+c^2+16=0$.

The numerical value of $a^{2021}+b^{2021}+c^{2021}$ is _____.

Solution:

$a-b=8, ab+c^2+16=0$

$\therefore b=a-8 \quad \therefore a(a-8)+c^2+16=0$

$\Rightarrow a^2-8a+16+c^2=0$

$\Rightarrow (a-4)^2+c^2=0$

Since a and c are real, a=4 and c=0 and a-b=8 $\Rightarrow b=-4$

$\therefore a^{2021}+b^{2021}+c^{2021}=4^{2021}+(-4)^{2021}+0^{2021}=0.$ Ans:0

30) The value of $\left(\frac{a}{a^2-4}-\frac{8}{a^2+2a}\right)\left(\frac{a^2-2a}{4-a}\right)+\frac{a+8}{a+2}$ is given to be 1.Then the value of 'a' is _____.

Solution:

$$\left(\frac{a}{a^2-4}-\frac{8}{a^2+2a}\right)\left(\frac{a^2-2a}{4-a}\right)+\frac{a+8}{a+2}=1$$

$$\Rightarrow\left(\frac{a}{(a-2)(a+2)}-\frac{8}{a(a+2)}\right)\left(\frac{a(a-2)}{4-a}\right)+\frac{a+8}{a+2}=1$$

$$\Rightarrow\left(\frac{1}{a+2}\right)\left[\frac{a}{a-2}-\frac{8}{a}\right]\left[\frac{a(a-2)}{4-a}\right]+\frac{a+8}{a+2}=1$$

$$\Rightarrow\left(\frac{1}{a+2}\right)\left[\frac{a^2-8a+16}{a(a-2)}\right]\left[\frac{a(a-2)}{4-a}\right]+\frac{a+8}{a+2}=1$$

$$\Rightarrow\left(\frac{1}{a+2}\right)\left[\frac{(4-a)^2}{a(a-2)}\right]\left[\frac{a(a-2)}{4-a}\right]+\frac{a+8}{a+2}=1$$

$$\Rightarrow\frac{4-a}{a+2}+\frac{a+8}{a+2}=1\Rightarrow 12=a+2\Rightarrow a=10$$ **Ans:10**

NMTC-53 -JUNIOR LEVEL -2022
SCREENING TEST - BASKARA CONTEST
(Std IX & X)

1. In an arithmetical progression, the n^{th} term is t_n. If $t_1 + t_3 = t_4 - 2$ and $t_6 + t_4 = t_{11} - 7$, then the value of t_{2021} is

   (a) 10103 (b) 100103 c) 1003 d) 100013

Solu : Let the first term = a and common difference = d.

$\therefore t_n = a + (n-1)d$

Now, $t_1+t_3=t_4-2 \Rightarrow a+a+2d = a+3d-2$

$\Rightarrow a-d = -2$ _____ (i)

Also $t_6+t_4=t_{11}-7 \Rightarrow a+5d+a+3d=a+10d-7$

$\Rightarrow a-2d = -7$ _____ (ii)

From (i) & (ii) d=5 and a=3

$\therefore t_{2021} = 3 + (2020)5 = 10103$ Ans. (a)

2. P, Q are the mid points of sides CA and CB respectively of a right triangle ABC, whose hypotenuse is AB, Then the numerical value of $\frac{AQ^2 + BP^2}{AB^2}$ is

   a) $\frac{4}{5}$ b) $\frac{3}{4}$ c) $\frac{5}{4}$ d) $\frac{1}{5}$

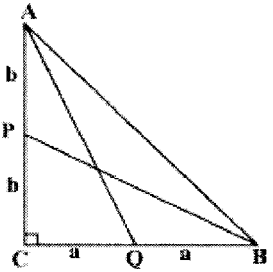
Solu :

Let CQ=QB = a, CP = PA = b

In Δ ACQ, $AQ^2=a^2+4b^2$ ___ (i)

In Δ PCB, $PB^2= b^2 + 4a^2$ __(ii)

In Δ ACB, $AB^2 = 4a^2 + 4b^2$ _____(iii)



$$\therefore \quad \frac{AQ^2+BP^2}{AB^2}=\frac{a^2+4b^2+b^2+4a^2}{4a^2+4b^2}=\frac{5\left(a^2+b^2\right)}{4\left(a^2+b^2\right)}=\frac{5}{4}$$

Ans (c)

3. If the zeroes of the polynomial $x^2 - px + q$ are in the ratio 2 : 3, then the value of $\frac{p^2}{q}$ is

a) $\frac{25}{6}$ b) $\frac{24}{5}$ c) $\frac{25}{3}$ d) $\frac{25}{7}$

Solu :

If the zeroes of the polynomial are α and β, then $\alpha + \beta = p$ and $\alpha\beta = q$,

Now $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$= p^2 - 4q$

$\Rightarrow \alpha - \beta = \sqrt{p^2-4q}$ ____(i)

Given $\frac{\alpha}{\beta} = \frac{2}{3}$

$$\Rightarrow \frac{\alpha+\beta}{\alpha-\beta}=\frac{5}{-1}\Rightarrow \frac{p}{\sqrt{p^2-4q}}=-5 \qquad \text{using (i)}$$

$$\Rightarrow p^2 = 25\,(p^2-4q) \Rightarrow \frac{p^2}{q} = \frac{100}{24} = \frac{25}{6} \quad \therefore \text{Ans (a)}$$

Alternatevely : $\frac{\alpha}{\beta} = \frac{2}{3} \Rightarrow \alpha = \frac{2\beta}{3}$

$$\therefore \alpha + \beta = -p \Rightarrow \frac{2\beta}{3} + \beta = -p \Rightarrow \frac{5\beta}{3} = -p$$

$$\Rightarrow p^2 = \frac{25\beta^2}{9} \quad \text{____ (i)}$$

and $\alpha\beta = q \Rightarrow \frac{2\beta^2}{3} = q$ _____ (i)

From (i) & (ii) $\frac{p^2}{q} = \frac{25\beta^2}{9} \times \frac{3}{2\beta^2} = \frac{25}{6}$ __Ans(a)

4. If $\tan\theta - \text{Cot}\theta = m$ and $\text{Cos}\theta + \text{Sin}\theta = n$, then the value of $(m^2+4)(n^2-1)^2$ is

a) 1 b) 2 c) 3 d) 4

Solu : $\tan\theta - \text{Cot}\theta = m$

$$\Rightarrow \frac{\text{Sin}\theta}{\text{Cos}\theta} - \frac{\cos\theta}{\sin\theta} = m$$

$$\Rightarrow \frac{\text{Sin}^2\theta - \cos^2\theta}{\text{Sin}\theta\text{Cos}\theta} = m$$

$$\therefore m^2 + 4 = \frac{\left(\text{Sin}^2\theta - \cos^2\theta\right)^2}{\text{Sin}^2\theta\text{Cos}^2\theta} + 4$$

$$\Rightarrow m^2 + 4 = \frac{\left(\text{Sin}^2\theta - \cos^2\theta\right)^2 + 4\text{Sin}^2\theta\text{Cos}^2\theta}{\text{Sin}^2\theta\text{Cos}^2\theta}$$

$$\Rightarrow m^2 + 4 = \frac{1}{\text{Sin}^2\theta\text{Cos}^2\theta} \quad \text{_____ (i)}$$

Again, $\text{Cos}\theta + \text{Sin}\theta = n$

$\Rightarrow 1 + 2\,\text{Sin}\theta\text{Cos}\theta = n^2$

$\Rightarrow n^2 - 1 = 2\,\text{Sin}\theta\text{Cos}\theta$

$\Rightarrow (n^2-1)^2 = 4\,\text{Sin}^2\theta\text{Cos}^2\theta$ _____ (ii)

Now $(m^2 + 4)(n^2-1)^2 = \frac{1}{\text{Sin}^2\theta\text{Cos}^2\theta} \times 4\,\text{Sin}^2\theta\text{Cos}^2\theta = 4$

Ans (d)

Alternatively, one can obtain the result (1) above as follows:

$$\tan\theta - \text{Cot}\theta = m$$

$$\Rightarrow \tan^2\theta - 2 + \cot^2\theta = m^2$$

$$\Rightarrow \tan^2\theta + 2 + \cot^2\theta = m^2 + 4$$

$$\Rightarrow m^2 + 4 = (\tan\theta + \cot\theta)^2 = \left(\frac{1}{\text{Sin}\,\theta\,\text{Cos}\,\theta}\right)^2$$

5. If x_1, x_2 satisfy $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$, then $\frac{x_1}{x_2}$ is equal to

(a) $\frac{9}{4}$ (b) $\frac{4}{9}$ (c) $\frac{9}{4}$ or $\frac{4}{9}$ (d) 1

Solu :

Squaring both sides, we get

$$\frac{x}{1-x} + \frac{1-x}{x} + 2 = \frac{169}{36}$$

$$\Rightarrow \frac{x^2 + (1-x)^2}{x\,(1-x)} = \frac{169}{36} - 2 = \frac{97}{36}$$

$$\Rightarrow 36\,[\,2x^2 - 2x + 1\,] = 97\,(x - x^2)$$

$$\Rightarrow 169x^2 - 169x + 36 = 0$$

x_1 and x_2 satisfy this equation

$$\therefore x_1 + x_2 = 1, \quad x_1 x_2 = \frac{36}{169}$$

$$\Rightarrow x_1 - x_2 = \pm\frac{5}{13}$$

Taking $x_1 + x_2 = 1$ and $x_1 - x_2 = \frac{5}{13}$, we get

$$x_1 = \frac{9}{13} \text{ and } x_2 = \frac{4}{13} \Rightarrow \frac{x_1}{x_2} = \frac{9}{4}$$

Taking $x_1+x_2 = 1$ and $x_1 - x_2 = -\frac{5}{13}$, we get

$$x_1 = \frac{4}{13} \text{ and } x_2 = \frac{9}{13} \Rightarrow \frac{x_1}{x_2} = \frac{4}{9}$$

Ans (c)

6. If $\frac{1}{b+c} + \frac{1}{c+a} = \frac{2}{a+b}$, then the value of $\frac{a^2+b^2}{c^2}$ is

(a) 1 (b) 2 (c) 3 (d) 4

Solu : $\frac{1}{b+c} + \frac{1}{c+a} = \frac{2}{a+b}$

$\Rightarrow \frac{a+b+2c}{(b+c)(c+a)} = \frac{2}{a+b}$

$\Rightarrow$ (a+b+2c) (a+b) = 2(b+c) (c+a)

$\Rightarrow a^2 + ab + ab + b^2 + 2ca + 2bc = 2bc + 2ab + 2c^2 + 2ca$

$\Rightarrow a^2 + b^2 = 2c^2$

$\Rightarrow \frac{a^2+b^2}{c^2} = 2$ Ans (b)

7. The sum of the squares of the roots of the equation

$$\log_{(3-4x^2)}(9-16x^4) = 2 + \frac{1}{\log_2(3-4x^2)} \text{ is}$$

(a) 1 (b) 2 (c) 3 (d) $-\frac{1}{2}$

Solu : $\log_{(3-4x^2)} (3-4x^2)\ (3+4x^2) = 2 + \frac{1}{\log_2 (3-4x^2)}$

$\Rightarrow \quad 1 + \log_{(3-4x^2)} (3+4x^2) = 2 + \log_{(3-4x^2)} 2$

$\Rightarrow \quad \log_{(3-4x^2)} (3+4x^2) - \log_{(3-4x^2)} 2 = 1$

$\Rightarrow \quad \log_{(3-4x^2)} \frac{(3+4x^2)}{2} = 1$

$\Rightarrow \quad 3 - 4x^2 = \frac{3 + 4x^2}{2}$

$\Rightarrow \quad 6 - 8x^2 = 3 + 4x^2 \Rightarrow 12x^2 = 3 \Rightarrow x^2 = \frac{1}{4}$

$\therefore \quad x_1 = \frac{1}{2}, \; x_2 = \frac{-1}{2} \qquad \therefore \; x_1^2 + x_1^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Ans (d)

8. The number of natural numbers 'n' such that $(n^3-8n^2+20n-13)$ would be a prime is

(a) 1 (b) 2 (c) 3 (d) 4

Solu : For n=1, the expression = o

$\therefore \; n^3-8n^2+20n-13 = (n-1)(n^2 - 7n+13)$

When $n-1 = 1$, $n^2 - 7n + 13$ is a prime

if $n^2 - 7n + 13 = 1$ then $n = 3, 4$

When $n = 3$, the expression = 2

and $n = 4$, the expression = 3

$\therefore$ For $n = 2, 3$ & 4, the expression is a prime.

Ans (c)

9. If x is real and $x^5 - x^3+x=a$, then the minimum value of x^6 is

(a) 2a (b) 2a-1 (c) 2a+1 (d) 3a

Solu : $x^5 - x^3 + x = a$

$\Rightarrow \quad x(x^4 - x^2 + 1) = a$ _____ (i)

Now $x, \; (x-1)^2 \geq 0$

$\Rightarrow \quad x^2 + 1 \geq 2x$ _____(ii)

Hence, $x^6 + 1 = (x^2)^3 + (1)^3 = (x^2 + 1)(x^4 - x^2+1)$

$\geq 2x(x^4 - x^2+1)$ [by (ii)]

$= 2a$ [by (i)]

$\therefore\ x^6 + 1 \geq 2a \quad \therefore\ x^6 \geq 2a-1 \quad \therefore$ Ans (b)

10. The maximum number of right angles which can occur among the interior angles of a convex polygon of n sides, where $n>4$ is

(a) 2 (b) 3 (c) 4 (d) 5

Solu :

In the case of a triangle, only one right angle is possible. In a qudrilateral, 4 right angles are possible (square or rectangle). Assume a convex polygon of side $n>4$.

Let 'k' interior angles be right angles. Each of the remaining interior angle must be less than two right angles.

Sum of all interior angles of a n-gon

$= (2n-4)$ right angles.

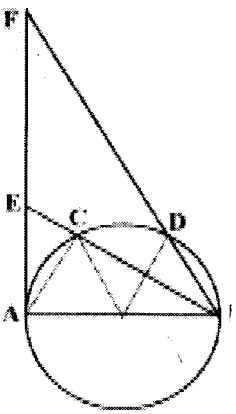
$\therefore \quad 2n-4 < k + 2(n-k)$

$\Rightarrow \quad k < 4$

$\therefore$ Maximum number of right angles = 3 Ans (b)

11. In the adjoining diagram CD is paralled to the diameter AB of the circle of radius 5 units. Line segment AEF is tangent to the circle. The value of AE . AF (in square units) is

(a) 25 (b) 50 (c) 100 (d) 200



Solu :

Let $\angle ABF = \alpha$ and $\angle ABC = \beta$

$\therefore \quad \angle BAC = 90^0 - \beta$

$\therefore \quad \angle ACO = 90^0 - \beta \ (\therefore AO = CO)$

$\angle OCB = \beta \quad (\therefore OB = OC)$

$\angle BCD = \beta$ ($\therefore$ CDIIAB, BC transversal)

$\therefore$ $\angle ACD = 90^0 - \beta + \beta + \beta$

$\therefore$ $\angle ACD = 90^0 + \beta$ and $\angle ABD = \alpha$

As ABCD is cyclic, $90^0 + \beta + \alpha = 180^0$

$\Rightarrow \alpha + \beta = 90^0$

$\therefore \angle BAC = \alpha$, $\angle BFA = 90^0 - \alpha = \beta$

$\Delta BAE \sim \Delta BCA \Rightarrow \frac{BA}{BC} = \frac{AE}{CA} \Rightarrow \frac{10}{BC} = \frac{AE}{CA} \Rightarrow AE = 10\,\frac{CA}{BC}$ ___(i)

$\Delta FAB \sim \Delta BCA \Rightarrow \frac{AB}{CA} = \frac{FA}{BC} \Rightarrow \frac{10}{CA} = \frac{FA}{BC} \Rightarrow FA = 10\,\frac{BC}{CA}$ ____ (ii)

From (i) and (ii) AE. AF = 100 Ans (c)

12. If $\frac{xy}{x+y} = \frac{4}{3} \Rightarrow \frac{x+y}{xy} = \frac{3}{4} \Rightarrow \frac{1}{y} + \frac{1}{x} = \frac{3}{4}$ then

(a) x + y = z (b) y + z = x (c) z + x = y (d) xyz = 1

Solu : $\frac{xy}{x+y} = \frac{4}{3} \Rightarrow \frac{x+y}{xy} = \frac{3}{4} \Rightarrow \frac{1}{y} + \frac{1}{x} = \frac{3}{4}$ _____(i)

Similarly $\frac{1}{y} + \frac{1}{z} = \frac{5}{12}$ _____(ii)

and $\frac{1}{z} + \frac{1}{x} = \frac{2}{3}$ ____ (iii)

Taking (i) - (ii) + (iii) gives $\frac{2}{x} = \frac{3}{4} - \frac{5}{12} + \frac{2}{3} = 1$

$\therefore x = 2$

Similarly we get y = 4 & z=6

$\therefore$ x+y=z Ans (a)

13. If $f(x) = x^{15} - 2022\,x^{14} + 2022\,x^{13} - 2022\,x^{12} + ... + 2022x$, then the value of $f(2021)$ is

(a) $(2021)^7$ (b) $(2021)^6$ (c) 0 (d) 2021

Solu : $f(x) = x^{15} - 2022.x^{14} + 2022.x^{13} - 2022.x^{12} + ... + 2022x$

$\therefore f(2021) = (2021)^{15} - (2021+1).(2021)^{14} + (2021+1)(2021)^{13} - (2021+1)(2021)^{12} + ... + (2021+1)(2021)$

$= (2021)^{15} - (2021)^{15} - (2021)^{14} + (2021)^{14} + (2021)^{13} - (2021)^{13} - (2021)^{12} + .. - .. + (2021)^2 + 2021$

$= 2021$ Ans (d)

14. The sum of the squares of all the roots of the equation $2^{x^2-1} - 3^{x^2} = 3^{x^2-1} - 2^{x^2+2}$ is

(a) 6 (b) 5 (c) $2\sqrt{3}$ (d) 9

Solu : $2^{x^2-1} - 3^{x^2} = 3^{x^2-1} - 2^{x^2+2}$

$\therefore \frac{2^{x^2}}{2} - 3^{x^2} = \frac{3^{x^2}}{3} - 2^{x^2}.4$

$\therefore \frac{2^{x^2}}{2} + 2^{x^2}.4 = \frac{3^{x^2}}{3} + 3^{x^2}$

$\therefore 2^{x^2} \times \frac{9}{2} = 3^{x^2} \times \frac{4}{3}$

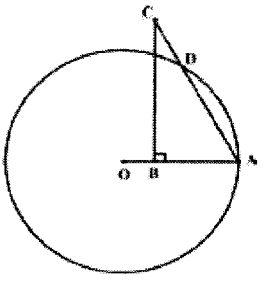
$\therefore \left(\frac{2}{3}\right)^{x^2} = \frac{8}{27} = \left(\frac{2}{3}\right)^3 \quad \therefore x^2 = 3$

$\therefore x_1 = \sqrt{3}, \; x2 = -\sqrt{3}$

$\therefore x_1^2 + x_2^2 = 3 + 3 = 6$ Ans (a)

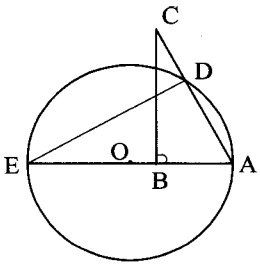
15. In the adjoining figure, O is the centre of the circle, CB is perpendicular to OA, CA cuts the circle in D.

The value of $\frac{CA.AD}{OA.AB}$ is



(a) 1 (b) 2 (c) 3 (d) $\frac{1}{2}$

Solu : Extend AO to intersect the circle in E. Join ED.



$\Delta ABC \sim \Delta ADE$

$\therefore \frac{AB}{AD} = \frac{AC}{AE}$

$\Rightarrow \quad AC.AD = AB.AE$

$\Rightarrow \quad AC.AD = AB.2\,OA$

$\Rightarrow \quad \frac{AC.AD}{OA.AB} = 2$

Ans (b)

Fill in the blanks

16. The roots of $(x+1)^{\log(x+1)} = 100(x+1)$ are x_1 and x_2 with $x_1 < x_2$. The value of $10x_1 + x_2$ is_____.

Solu : Let $x+1 = a$

Given equation is $a^{\log a} = 100a$

$\log (a^{\log a}) = \log 100 + \log a$

$(\log a)^2 - \log a - 2 = 0$

Let $y = \log a$ then

Eq. is $y^2 - y - 2 = 0$

$\Rightarrow \quad y = 2$ or $y = -1$

$\Rightarrow \quad \log a = 2$ or $\log a = -1$

$\Rightarrow \quad a = 100$ or $a = \frac{1}{10}$

$\Rightarrow \quad x+1 = 100$ or $x+1 = \frac{1}{10}$

$\Rightarrow \quad x_2 = 99$ or $x_1 = -\frac{9}{10}$

$\therefore \quad 10x_1 + x_2 = -9 + 99 = 90$ Ans : 90

17. The numerical value of

$\frac{1}{x^3}\{(x+5y-3z)^3+(x-5y+3z)^3+6x(x+5y-3z)(x-5y+3z)\}$ is__

Solu : Let $a=x+5y-3z$ and $b=x-5y+3z$

$\therefore (a+b)=2x$

Now, $(x+5y-3z)^3+(x-5y+3z)^3+6x(x+5y-3z)(x-5y+3z)$

$=a^3+b^3+3(2x)ab$

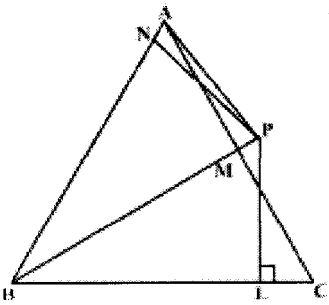
$=a^3+b^3+3ab(a+b)$

$=(a+b)^3$

$=8x^3$ $\quad\therefore$ Required Value $=\frac{1}{x^3}\{8x^3\}$

$=8$ Ans. 8

18. P is a point outside equilateral ΔABC, but inside the region $\angle ABC$. If the distances from P to BC, CA and AB are h_1, h_2 and h_3 resp., and $h_1-h_2+h_3=6$ and the area of ΔABC is $\lambda\sqrt{3}$ the value of λ is_____.



Solu : ΔABC is equlateral Let $PL \perp BC$, $PM \perp CA$, $PN \perp AB$ and so $PL=h_1$, $PM=h_2$, $PN=h_3$ as given. Join PA, PB, PC. Let $AB=BC=CA=a$. ___ (i)

Also given $h_1-h_2+h_3=6$____(ii)

Now, Area of ABC = Area (ΔAPB)+Area (ΔBPC) - Area (ΔAPC)

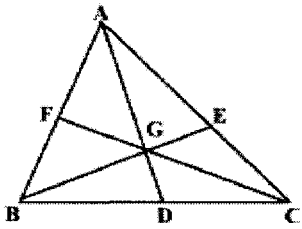
$\therefore \lambda\sqrt{3}=\frac{1}{2}ah_3+\frac{1}{2}a.h_1-\frac{1}{2}a.h_2$

$\therefore \lambda\sqrt{3}=\frac{1}{2}a.6=\frac{1}{2}a.\frac{\sqrt{3}a}{2} \Rightarrow a=4\sqrt{3}$

$\therefore \lambda\sqrt{3}=\frac{1}{2}48.\frac{\sqrt{3}}{2} \Rightarrow \lambda=12$ Ans $\lambda=12$

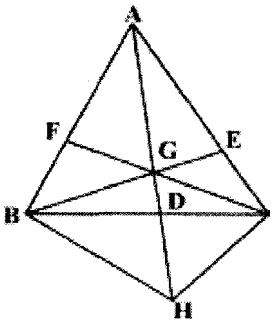
19. Given G is the centroid of ΔABC. GA=$2\sqrt{3}$, GB = $2\sqrt{2}$ and GC = 2. If the area of Δ ABC is $\lambda\sqrt{2}$, then the value of λ is____.

Solu :



From the figure, it is clear that as BG, GC and GD are given, one can obtain the length of BC from Δ GBC. Similarly AC and AB can be calculated and by Heron's formula, area can be obtained to be compared with the given area.

Another elegant solution is given (by the paper setter) is as follows :



Extend AD to H so that GD=DH=$\sqrt{3}$ and GH = $2\sqrt{3}$. BG = $2\sqrt{2}$ and BH = GC = 2 Here, $BG^2 + BH^2 = 8 + 4 = 12 = GH^2$

Hence $\angle GBH = 90^0$ and GBHC is a rectangle.
Area (GBHC) = BGxBH = $2\sqrt{2} \times 2 = 4\sqrt{2}$

$\therefore$ Area (Δ GBC) = $2\sqrt{2}$.

As $\frac{AD}{GD} = \frac{3}{1}$, Area (ABC) = $3(2\sqrt{2}) = 6\sqrt{2}$

$\therefore \lambda = 6$ Ans.

20. The number of pairs of natural numbers (m, n) such that $(m^2+n) - (m+n^2) = 10$ is_____.

Solu : $(m^2+n) - (m + n^2) = 10$

$\Rightarrow$ $m^2 - n^2 - m + n = 10$

$\Rightarrow$ $(m - n)(m + n - 1) = 10$

Case - 1 Let $10 = 1 \times 10$

$\therefore m - n = 1 \Rightarrow m = n + 1 \;\&\; m + n - 1 = 10$

$\Rightarrow n + 1 + n - 1 = 10$

$\Rightarrow n = 5 \;\&\; m = 6 \quad \therefore (6, 5)$

Case - 2 Let $10 = 2 \times 5$

$\therefore m - n = 2 \Rightarrow m = n + 2 \;\&\; m - n - 1 = 5$

$\Rightarrow n + 2 + n - 1 = 5$

$\Rightarrow n = 2 \;\&\; m = 4 \quad \therefore (4, 2)$

By taking $10 = 10 \times 1$ and $10 = 5 \times 2$, we get -ve values of m & n. $\therefore$ Possible pairs are 2 Ans : 2

21. The number of real values of n for which $\sqrt{120 - \sqrt{n}}$ is an integer is_____.

Solu : $\sqrt{120 - \sqrt{n}}$ is an integer

$\Rightarrow 120 > \sqrt{n}$ and n be a positive integer which is a perfect square.

As there are 11 perfect square numbers (0,1,4,9,....100) less than 120, there will be 11 perfect square integer values of n starting from 100 to 14400 for which the given condition is satisfied. Ans 11

22. The number of cuboids with unequal integer side lengths in arithmetic progression such that the volume is less than 25 cubic units is_____.

Solu : If a, b and c are the side lengths of the cuboid then

$a \times b \times c < 25$; $a \neq b \neq c$; a, b, c are integers and are in A.P.

Take

$a=1, \quad b=2, \quad c=3 \quad \Rightarrow \quad abc = 6$ ✓

$a=1, \quad b=3, \quad c=5 \quad \Rightarrow \quad abc = 15$ ✓

$a=1, \quad b=4, \quad c=7 \quad \Rightarrow \quad abc = 28$ ×

$a=2, \quad b=3, \quad c=4 \quad \Rightarrow \quad abc = 24$ ✓

Now further for any values of a, b, c, the volume will be more than 25.

$\therefore$ The possible number of cuboids is 3 Ans : 3.

23. If a and b are real such that $a^2+b^2+8a - 14b+65=0$, then the numerical value of (a^2+b^2) is_____.

Solu :

$a^2 + b^2+8a - 14b + 65 = 0$

$a^2 + 8a + 16 + b^2-14b + 49 = 0$

$(a + 4)^2 + (b - 7)^2 = 0$

$a = -4, \; b=7$

$a^2 + b^2 = 16 + 49 = 65$ Ans : 65

24. If a is real and $a^4+a^3+a^2+a+1 = 0$ then the numerical value of $a^{2022} + \frac{1}{a^{2022}}$ is____ .

Solu : Clearly, a=1 does not satisfy the equation. Hence $a \neq 1$ and $a - 1 \neq 0$. Multiplying by $(a - 1)$

$(a - 1)(a^4+a^3+a^2+a+1) = 0$

$\Rightarrow a^5-1=0 \quad \Rightarrow a^5=1$

Now $a^{2022} + \frac{1}{a^{2022}} = a^{2020}a^2 + \frac{1}{a^{2020}.a^2} = (a^5)^{404}.a^2 + \frac{1}{(a^5)^{404}.a^2}$

$= a^2 + \frac{1}{a^2}$

Note : In the example, the condition given that "a is real" is not correct. The given equation reduces to $a^2 + \frac{1}{a^2}$

25. If a=1 and b, c, d > o are reals such that $a^4 + b^4 + c^4 + d^4 = 4abcd$, then the numerical value of $a^{2022} + b^{2022} + c^{2022} + d^{2022}$ is____.

Solu : For the numbers $a^4 + b^4 + c^4 + d^4$, by AM ≥ GM inequality,

$$\frac{a^4 + b^4 + c^4 + d^4}{4} \geq \sqrt[4]{a^4 . b^4 . c^4 . d^4}$$

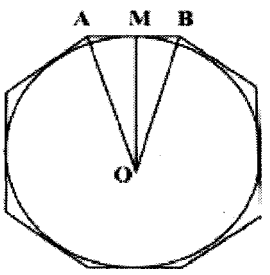
$\therefore$ $a^4 + b^4 + c^4 + d^4 \geq 4.abcd$

Given a = 1, $1 + b^4 + c^4 + d^4 \geq 4\,bcd$

but equality is given, which will be possible only for b=c=d=1.

$a^{2022} + b^{2022} + c^{2022} + d^{2022} = 4$ - Ans. 4

26. For Octagon : Area - A / Perimeter = P ; for circle: Radius = R
From the diagram, $AB = \frac{P}{8}$, OM = R and Area of $OAB = \frac{A}{8}$

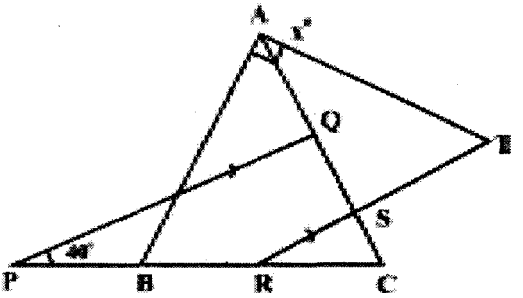


$\therefore$ Area of $OAB = \frac{1}{2}$ OA.OM

$\therefore \frac{A}{8} = \frac{1}{2} \cdot \frac{P}{8} \cdot R$

$\Rightarrow \frac{RP}{A} = 2$ Ans. 2

27. In the adjoining figure, ABC is an equilateral triangle - PQ and RST are parallel and AS = ST. Also $\angle CPQ = 40^0$ and $\angle BAT = x^\circ$. Then x is____



Solu : $\angle SRC = \angle QPC = 40^\circ$ and $C=60^0$

$\therefore \angle RSC = 80^0 = \angle AST$ _____ (i)

$\angle SAT = x\text{-}60 = \angle ATS$ $(\because AS=ST)$____ (ii)

From (i) & (ii) $2(x\text{-}60^0) + 80^0 = 180^0 \Rightarrow x = 110^0$ Ans.

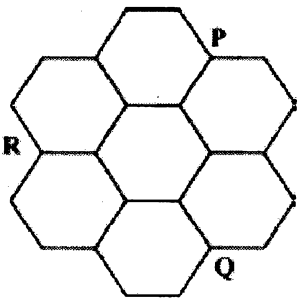
28. The n^{th} term of a sequence is given by

$t_n = \dfrac{\left[1+(-1)^n\right]+1}{3n+1}$. Then the value of $\dfrac{1}{t_1}+\dfrac{1}{t_3}+\dfrac{1}{t_5} =$ ___

Solu : $t_1 - \dfrac{1}{4}$, $t_3 = \dfrac{1}{10}$, $t_5 = \dfrac{1}{16}$

$\therefore \dfrac{1}{t_1}+\dfrac{1}{t_3}+\dfrac{1}{t_5} = 4+10+16 = 30$ Ans

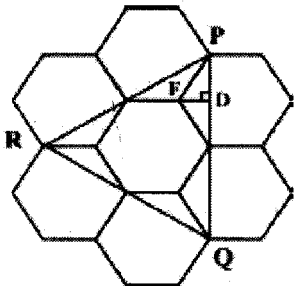
29. Seven regular hexagons of equal sides of 1 unit are placed as in the figure. The are of ΔPQR (in square unit) is $a\sqrt{3}$. Then a is_____.



Solu : From the figure,

Δ FDP is a 30^0 - 60^0 - 90^0 triangle

with hypo. FP=1 $\therefore PD = \dfrac{\sqrt{3}}{2}$,



PQ=$2\sqrt{3}$ Δ PQR is equilateral.

$\therefore$ Area $= \frac{\sqrt{3}}{4}\left(2\sqrt{3}\right)^2 = 3\sqrt{3}$

$\therefore$ a = 3 Ans

30. Two real numbers m, n ($m \neq 0, n \neq 0$) are such that mn = m – n.

The numerical value of $\frac{m}{n}+\frac{n}{m}-mn$ is____.

Solu : $\frac{m}{n}+\frac{n}{m}-mn \Rightarrow \frac{m^2+n^2}{mn}-mn$

$= \frac{m^2+n^2-(mn)^2}{mn} = \frac{m^2+n^2-(m-n)^2}{mn}$

$= 2$ Ans 2

NMTC-53 - INTER -2022
SCREENING TEST - RAMANUJAN CONTEST
(Std XI & XII)

1. Given A

$$\frac{\frac{1}{2}}{1+\frac{1}{2}=}+\frac{\frac{1}{3}}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)}+....+\frac{\frac{1}{2021}}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{2021}\right)}$$

The value of [A] where [A] denotes the greatest integer not exceeding A is

(a) 1 (b) 2021 (c) 0 (d) 2

Solu : Let us add and substract 1 in the given sum A.

$$\therefore A=1+\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{3}}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)}+....+\frac{\frac{1}{2021}}{\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{2021}\right)}-1$$

$$=\left[\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+.....\right]-1$$

Hence, the denominators 1, 3, 6, 10,.... determine a series whose $t_n=\frac{2}{n(n+1)}$

$$\therefore \quad A=\left[\sum \frac{2}{n(n+1)}\right]-1$$

$$=2\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)-1$$

$$=2\left[1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+.....+\frac{1}{n+1}\right]-1$$

$$=2\left[1-\frac{1}{n+1}\right]-1$$

$= \frac{2n}{n+1} - 1$

$= \frac{n-1}{n+1}$

Taking $n = 2021$ we get $A = \frac{2020}{2021} < 1$

$\therefore [A] = 0$ Ans (a)

2. If α, β are the roots of $x^2+7x+3=0$ and $\frac{2\alpha}{3-4\alpha}, \frac{2\beta}{3-4\beta}$ are the roots of $ax^2 + bx + c = 0$ and a,b,c have no common factors, then $a + b + c$ is equal to

(a) 71 (b) 61 (c) 81 (d) 91

Solu: Let $\frac{2\alpha}{3-4\alpha} = y \Rightarrow \alpha = \frac{3y}{2+4y}$

$\Rightarrow \left(\frac{3y}{2+4y}\right)^2 + 7\left(\frac{3y}{2+4y}\right) + 3=0$

$\Rightarrow 47y^2+30y+4=0$

y is a root of $ax^2+bx+c = 0$

$\Rightarrow a = 47, b = 30, c = 4$

$\Rightarrow a + b + c = 81.$ Ans (c)

3. Let t_n be the number of possible triangles formed b y joining the vertices of an n-sided regular polygon. If $t_{n+1} - t_n = 21$, then the polygon is a

(a) Hexagon (b) Septagon (c) Octagon (d) Nonagon

Solu : $t_n = {}^nC_3 = \frac{n!}{6\,(n-3)!} = \frac{n(n-1)(n-2)}{6}$

$$t_{n+1} = \frac{(n+1)(n)(n-1)}{6}$$

By given data,

$$t_{n+1} - t_n = 21$$

$$\Rightarrow \frac{(n+1)(n)(n-1)}{6} - \frac{n(n-1)(n-2)}{6} = 21$$

$$\Rightarrow \frac{n(n-1)}{6}[(n+1-(n-2)] = 21$$

$$\Rightarrow \frac{n(n-1)}{2} = 21 \Rightarrow n(n-1) = 42$$

$$\Rightarrow n = 7$$ Ans (b)

4. The sum of all the roots of the following equations solved simultaneously

$$\frac{1}{x+2y} + y = 2, \quad \frac{y}{x+2y} = -3 \text{ is}$$

(a) $-\frac{8}{3}$ (b) $\frac{8}{3}$ (c) $\frac{7}{3}$ (d) $\frac{-3}{8}$

Solu : $\frac{1}{x+2y} + y = 2$

$\Rightarrow 1 + xy + 2y^2 = 2x+4y$

$\Rightarrow 2y^2+xy-2x-4y+1=0$

$\Rightarrow y(x+2y) - 2(x+2y) + 1 = 0$

$\Rightarrow (x+2y)(y-2) + 1 = 0$ _____(i)

$$\frac{y}{x+2y} = -3 \Rightarrow y = -3x - 6y$$

$$\Rightarrow 7y = -3x \Rightarrow y = \frac{-3x}{7}$$ _____ (ii)

Using (ii) in (i),

$$\left(x-\frac{6x}{7}\right)\left(\frac{-3x}{7}-2\right)+1=0$$

$$\Rightarrow \quad \left(\frac{x}{7}\right)\left(\frac{-3x-14}{7}\right)+1=0$$

$\Rightarrow \quad -3x^2-14x+49=0$

$\Rightarrow \quad 3x^2+14x-49=0$

$\Rightarrow \quad 3x^2+21x-7x-49=0$

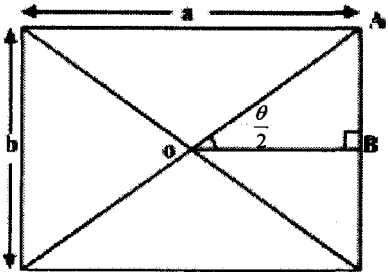
$\Rightarrow \quad (x+7)(3x-7)=0 \quad x=-7 \text{ or } x=7/3$

From (ii) $y=3$ or $y=-1$

$\therefore$ Sum of all roots $= -8/3$ (Ans (a)

5. The perimater of a rectangle is 2x and the area is $\frac{3x^2}{16}$. The acute angle between the diagonals of the rectangle is $2\tan^{-1}\frac{1}{p}$; then the value of p is

(a) 2 (b) 3 (c) 4 (d) 5



Solu :

If the length of the rectangle is 'a' and breath is 'b' them by given data, $p=2x$ and $A=\frac{3x^2}{16}$

$\therefore \quad 2(a+b)=2x \Rightarrow a+b=x$ ____ (i)

and $ab=\frac{3x^2}{16}$

$\therefore$ Using the indentity $(a-b)^2=(a+b)^2-4ab$, we get

$$a-b=\frac{x}{2} \quad \text{_____(ii)}$$

Solving (i) & (ii) $\quad a=\frac{3}{4}x, \quad b=\frac{1}{4}x$

From the figure OB = $\frac{3}{8}x$ and AB = $\frac{1}{8}x$ and $\angle AOB = \theta/2$.

$\therefore \tan \theta/2 = \frac{AB}{OB} = \frac{1}{3} \Rightarrow \theta = 2 \tan^{-1} \frac{1}{3}$

$\therefore p = 3$ Ans (b)

6. For all values of x, if $(k-2)x^2+8x+(k+4)$ is positive, then the least positive integral value of 'k' is

(a) an even integer (b) a prime number

(c) a multiple of 7 (d) an irrational number

Solu : Given $(k-2)x^2+8x+k+4$ positive, it is clear that for $k>2$, the discriminant must be negative i.e.

$b^2 - 4ac < o$

$\Rightarrow 64-4(k-2)(k+4) < o$

$\Rightarrow 16-(k-2)(k+4) < o$

$\Rightarrow -k^2-2k+24 < o$

$\Rightarrow k^2+2k-24 > o$

$\Rightarrow (k+6)(k-4) > o$

$\therefore$ The least positive integral value of k must be 4. Ans. (a)

7. The number of integer triples (a, b, c) that satisfy the condition that the sum of the squares of the first two is 3 more than the square of the third is

(a) 0 (b) 2 (c) 3 (d) infuite

Solu : $a^2 + b^2 = c^2+3$

Take two numbers 2a, 2a+1

Let us search the value of x such that

$x + 4a^2 = (2a+1)^2 + 3$

$\therefore\ x = (2a+1)^2 - 4a^2+3$

$\therefore\ x = 4\,(a+1)$

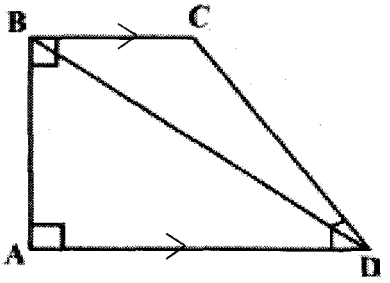
x is a perfact square. $\therefore$ a+1 must be a perfact square.

$\therefore$ There are infinite values so that x is a perfact square.

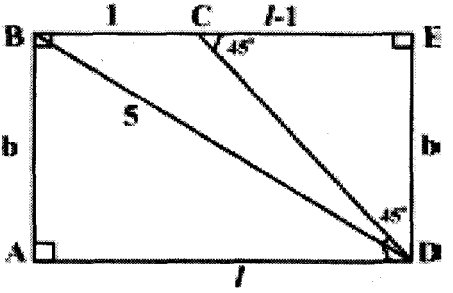
$\therefore$ The number of triples is infinite. Ans(d)

8. In the adjoining figure $\angle ADC = 45°$, BD=5 units and BC=1 unit. The area of the trapezoid ABCD in square units is

(a) $\frac{13}{2}$ (b) $\frac{15}{2}$ (c) $\frac{17}{2}$ (d) $\frac{19}{2}$



Solu :



Extend BC and Draw DE $\perp$ it to get E. let AD $= \ell$, AB=b

$\therefore$ CE $= \ell - 1$ and DE=b.

$\angle CDE = \angle DCE = 45^0$ hence $b = \ell - 1$ _____ (i)

From ΔABD, $\ell^2 + b^2 = 25$ _____ (ii)

Using (i) in (ii), we get $\ell = 4$ and $b = 3$

$\therefore$ Area of ABCD $= \frac{1}{2}$ AB (BC+AD)

$= \frac{1}{2} b (1 + \ell)$

$= \frac{1}{2} .3 (5) = \frac{15}{2}$ Ans (b)

9. If $\alpha, \beta, \gamma, \delta,$ are the roots of the equation $x^4-7x^3+8x^2-5x+10=0$, then the value of $(\alpha^2+2)(\beta^2+2)(\gamma^2+2)(\delta^2+2)$ is

(a) 273 (b) 216 (c) 173 (d) 166

Solu : Let $x^4-7x^3+8x^2 - 5x+10$

$= (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

Taking $x = -i\sqrt{2}$ we get

$(-2 + 9\sqrt{2}i) = (i\sqrt{2} - \alpha)(i\sqrt{2}-\beta)(i\sqrt{2}-\gamma)(i\sqrt{2}-\delta)$ _____(i)

Taking $x = i\sqrt{2}$ we get

$(-2- 9\sqrt{2}i) = (i\sqrt{2}-\alpha)(-i\sqrt{2}-\beta)(-i\sqrt{2}-\gamma)(-i\sqrt{2}-\delta)$ _____(ii)

Multiplying (i) & (ii) we get

$(\alpha^2+2)(\beta^2+2)(\gamma^2+2)(\delta^2+2) = 166.$ Ans(d)

10. The equations $x^2-x+m=0$ and $x^2-x+3m=0$ are such that a root of the second equation is double of one of the roots of the first equation. Then m is a root of the equation _____.

(a) $x^2+15x+36=0$ (b) $3x^2-17x+22=0$

(c) $4x^2-13x-17=0$ (d) $18x^2+31x-10=0$

Solu : Let the roots of the equation x^2-x+m=0 be α and β.

Here Δ = 1-4m and $\alpha = \frac{1+\sqrt{1-4m}}{2}$ ____ (i)

Let the roots of the equation x^2-x+3m=0 be 2α and β^1

Here Δ = 1-12m and $2\alpha = \frac{1+\sqrt{1-12m}}{2}$ ____(ii)

From (i) and (ii)

$$1+\sqrt{1-4m} = \frac{1+\sqrt{1-12m}}{2}$$

$$\Rightarrow 2+2\sqrt{1-4m} = 1+\sqrt{1-12m}$$

On simplifying we get m = 0 or m = –2 Clearly $m \neq o$

$\therefore$ m = -2 and this value satisfies eg. $18x^2+31x-10-0$

Ans (d)

11. In a scalene triangle, S is the circmcentence and I, I_1, I_2, I_3 are the incentre and ex-centres resp. The circum radius of the triangle is R. Then the value of

R^{-2} ($SI^2+SI_1^2+SI_2^2+SI_3^2$) is____.

(a) 8 (b) 9 (c) 11 (d) 12

Solu :

We know that

$SI^2 = R^2-2Rr$, $SI_1^2, = R^2+2Rr_1$, $SI_2^2 = R^2+2Rr_2$

$SI_3^2=R^2+2Rr_3$ Where r, r_1, r_2, r_3 are the radii of the incircle and excircles.

Now R^{-2} ($SI^2+SI^2_1, +SI^2_2+SI^2_3$)

$= R^{-2}$ [$R^2 - 2Rr + R^2+2Rr_1 + R^2+2Rr_2+R^2 + 2Rr_3$]

$= 4-\frac{2r}{R}+\frac{2r_1}{R}+\frac{2r_2}{R}+\frac{2r_3}{R}$

$$= 4 + \frac{2}{R}\ (r_1 + r_2 + r_3 - r)$$

$$= 4 + \frac{2}{R}\ (4R)$$

$$= 12$$ Ans (d)

12. There are 8 greeting cards, each of a different colour and 8 envelopes of the same eight colours. The number of ways in which the cards can be put in the envelopes such that exacely four of the cards go into the envelope of the right colour is

(a) 70 (b) 192 (c) 280 (d) 630

Solu : There are 9 ways to place 4 different cards into 4 different envelopes such that no card goes to the envelope of same colour. Choose 4 cards out of 8 in ${}^8C_4$ ways and put them in the envelopes of their respective colours.

The remaining 4 are placed so that none of them go to the envelope of the same colour. Thus the number of ways is

$$9\ x \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 630.$$ Ans (d)

13. A sequence $\{a_n\}$ with $a_1 = 29$ and $a_2 = 7$ satisfies (29.7) $(a_i^2 - a_{i-1}\ a_{i+1}) = a_i a_{i-1}$ for $i \geq 2$. The first value of k such that $a_k = 0$ is

(a) 40 (b) 43 (c) 47 (d) 51

Solu : We can rewrite the given relation as

$$\frac{a_i}{a_{i-1}} - \frac{a_{i+1}}{a_i} = \frac{1}{29.7}$$

Putting $t_{i-1} = \frac{a_i}{a_{i-1}}$, we can write

$$t_{i-1} - t_i = \frac{1}{29.7}$$

$$t_{i-2} - t_{i+} \ldots\ldots t_2 - t_3 = \frac{1}{29.7}$$

Adding the above, we get

$$t_2 - t_i = \frac{i-2}{29.7}$$

Also $t_2 = \frac{7}{29}$ Hence

$$t_i = \frac{7}{29} - \frac{i-2}{29.7} = \frac{51-i}{29.7}$$

Hence $t_i = 0$ when $i = 51$. Ans (d)

14. If $x_1, x_2, x_3, \ldots, x_n$ are positive integers such that $x_1+x_2+x_3+\ldots\ldots+x_n = 25$, then the maximum possible value of $x_1.x_2.x_3\ldots.x_n$ is

(a) 2187 (b) 4372 (c) 8748 (d) 13122

Solu :

Suppose that $x_1+x_2+\ldots\ldots+x_n=N$ and $x_1, x_2, \ldots x_n$ are positive integers. Let $N = 3K+r$, $r \in \{0,1,2\}$. If P(N) is the maximum product $x_1x_2x_3\ldots x_n$, then

$$P(N) = \begin{cases} 3^k \text{ if } r=0 \\ 4.3^{k-1} \text{ if } r=1 \\ 2.3^k \text{ if } r=2 \end{cases}$$

We can assume that $x_1 \geq x_2 \geq x_3 \geq \ldots\ldots \geq x_n$.

Note that (a-3)+3=a and (a-3).3>a when $a \geq 5$. Hence if we replace each $x_i \geq 5$ by x_i-3 and 3, we get a larger product without altering the sum. Applying this repeatedly we end up with $x_1 \leq 4$.

Replacing each occurence of 4 by two 2's changes neither the

sum nor the product, whereas combining each occurence of 1 with any other xi increases the product (Since $x_{i+1} > x_i$). Therefore we may further assume that each x_1 is either 3 or 2.

We claim that the number of 2's among the xi cannot exceed 2. Any set of three 2's can be replaced by two 3's - while maintaining the sum, the product is increased by doing this. Hence this claim.

If r =0, the number of 2's in the sum must be 0. hence $P(N)=3^k$.

If r =1, the number of 2's must be 2 and Hence $P(N)=4.3^{k-1}$.

If r = 2, there is a unique 2 in the sum and $P(N) = 2.3^{K-1}$.

Here, $25 = 8.3+1$ and hence the max.value is $4.3^7 = 8748$.

Ans (c)

15. In each round of a game two players compete, the one will win, the other will lose. The total score of each player is the number of games they win. In this game, every pair of players Saket, Samrudh, Achyut and Karthik compete in one round for a total of six rounds. Given that Saket wins his game against Karthik and after all rounds are completed Saket, Samrudh and Achyut have the same score. The score of Karthik is

(a) 0 (b) 1 (c) 2 (d) 3

Solu :

Since there are 6 games and each game has a winner, the total scores of all the four players equals 6. Since Saket won against Kathik, score of Saket is not zero and cannot be more than 2 (Since Saket, Samrudh and Achyut have the same score). If Saket, Samrudh and Achyut each have score 1, then Karthik's score must be 3. This means Karthik has won all the games which is not true. Hence Saket, Samrudh and Achyut have score 2 each and Karthik's score is zero.

Ans (a)

16. The number $N = \frac{2021! + 2020!}{2019!}$ is an integer.

The number of positive divisors of N is____

Solu : $\frac{2021! + 2020!}{2019!} = \frac{2020!\,(2021+1)}{2019!}$

$= 2020 \times 2022$

$= 2^3.3.5.37.101.$

Hence the number of factoir is 64.

17. The number of prime factors of $3^{18}-2^{18}$ is___

Solu : $3^{18}-2^{18} = (3^9-2^9)\,(3^9+2^9)$

$= (3^3-2^3)\,(3^6+3^3.2^3+2^6)\,(3^3+2^3)\,(3^6-3^3.2^3+2^6)$

$= (3-2)\,(3^2+3.2+2^2)\,(3^6+3^3.2^3+2^6)\,(3+2).$

$.(3^2-3.2+2^2)\,(3^6-3^3.2^3+2^6)$

$= 1.19.1009.5.7.577$

Since 19, 1009, 5, 7, 577 are all primes, the number of prime factors is 5.

18. Let m be the largest real solution of the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x1-19} = x^2-11x-4$$

There are positive integers a, b, c and d such that $m = a + \sqrt{b + \sqrt{c}}$ Then the value of a+b+c is____.

Solu :

The first step is to notice that the 4 on the right hand side can simplify the terms on the left hand side. If we distribute 1 to $\frac{3}{x-3}$, then the fraction becomes of the form $\frac{x}{x-3}$. A

similar cancellation happens with the other terms. If we assume x=o is not the highest solution (if we realise it is, we can always track back) we can cancel the common factor x from both sides of the equation.

$$\therefore \quad \frac{1}{x-3} + \frac{1}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x - 11$$

Then if we make the substituion y = x–11, we can further simplify as

$$\frac{1}{y+8} + \frac{1}{y+6} + \frac{1}{y-6} + \frac{1}{y-8} = y, \text{ if group}$$

and combine the terms of the form y-n and y+n, we get

$$\frac{2y}{y^2-64} + \frac{2y}{y^2-36} = y$$

Then we can cancel out a y from both the sides, knowing that x=11 is possible soultion. After we do that, we can make the final substitution $z=y^2$.

$$\frac{2}{z-64} + \frac{2}{z-36} = 1$$

$$\Rightarrow \quad 2z - 128 + 2z - 72 = (z-64)(z-36)$$

$$\Rightarrow \quad 4z-200 = z^2-100z + 64(36)$$

$$\Rightarrow \quad z^2-104z + 2504 = 0.$$

Using the quadratic formula, we get the largest solution for z is $52+10\sqrt{2}$. Then repeatedly substituting backwards, we find that the largest value of x is $11 + \sqrt{52+\sqrt{200}}$. The answer is thus 11+52+200 = 263.

19. In a regular hexagen ABCDEF, given BF = $2\sqrt[4]{12}$. The area of the pentagon BCDEF is_____.

Solu :

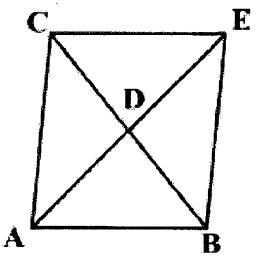
The area of the hexagon is twice the area of the equilateral triangle with the side length BF, The are of the hexagon is

$$= \frac{2\,x\,BF^2\sqrt{3}}{4} = \frac{\left(2\sqrt[4]{12}\right)^2\sqrt{3}}{2} = \frac{\left(4\sqrt{12}\right)\left(\sqrt{3}\right)}{2} = 12$$

The pentagon has 5/6 of the area of the hexagon and hence the required area is 10.

20. Two sides of a triangle have lengths 5 and 13. The median on the third side has length m. The sum of all possible integer values of m is___.

Solu :



Let AB = 5, AC=13 and AD be the median to the side BC.
Extend AD to E such that AE = 2AD and join EC, EB. Then ABEC is a parallelogram. From triangle ABE, we have, 5+13 >2m, 2m+5>13, 13+2m>5. This reduce to 4< m < 9 and m > –4, Thus the possible integer values of m are 5, 6, 7 and 8 and sum of these values is 26.

21. Let $a_n = 2 - \frac{1}{n^2 + \sqrt{n^4 + \frac{1}{4}}}$ for n = 1, 2,.....

The value of $\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_{20}}$ is____.

Solu : $a_k = 2 - \dfrac{1}{k^2 + \sqrt{k^4 + \frac{1}{4}}}$

$$= 2 - \frac{k^2 - \sqrt{k^4 + \frac{1}{4}}}{-1/4}$$

$$= 2 + 4k^2 - 2\sqrt{4k^4 + 1}$$

$$= (2k^2+2k+1) + (2k^2-2k+1) - 2 - 2\sqrt{(2k^2+2k+1)(2k^2-2k+1)}$$

$$= \left(\sqrt{2k^2+2k+1} - \sqrt{2k^2-2k+1}\right)^2$$

$$\sum_{k=1}^{20} \sqrt{a_k} = \sum_{k=1}^{20} \left(\sqrt{2k^2+2k+1} - \sqrt{2k^2-2k+1}\right)$$

Thus, $\left(\frac{100.99.98.97}{24}\right)$

$$= \sum_{k=1}^{20} \left(\sqrt{2(k+1)^2 - 2(k+1) + 1} - \sqrt{2k^2-2k+1}\right)$$

$$= \sqrt{2.21^2 - 2.21 + 1} - 1$$

$$= 29 - 1$$

$$= 28$$

22. If $f\left(\frac{2x-1}{x-3}\right) = 2x+5$ and $f(4) - f(5) = \frac{p}{q}$, where p,q are primes, then the value of (p+q) is___

Solu :

$\frac{2x-1}{x-3} = 4 \Rightarrow 2x-1=4x-12 \Rightarrow x = \frac{11}{2}$

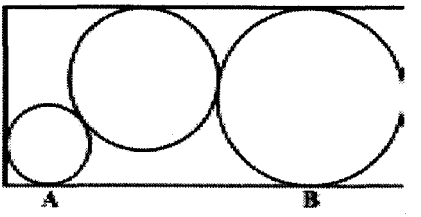
and $f(4)=2\left(\frac{11}{2}\right)+5=16.$

Again, $\frac{2x-1}{x-3}=5 \Rightarrow 2x-1=5x-15 \; x=\frac{14}{3}$

and $f(5)=2\left(\frac{14}{3}\right)+5=\frac{43}{3}$

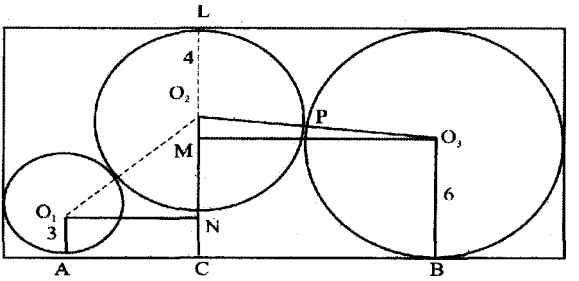
Now $f(4)-f(5)=16-\frac{43}{3}=5/3.$

$\therefore p=5, q=3 \quad \therefore$ $p+q=8.$



23. Three circles are arranged in a rectangle as shown in the figure, the radii being 3, 4 and 6. If the distance AB is $a\sqrt{a}$, the value of a is____

Solu :



Circles $(O_1, 3)$ and $(O_2, 4)$ touches each other in Q. Circles $(O_2, 4)$ and $(O_3, 6)$ touches each other in P. O_1A, O_2C and O_3 B are all perpendicular to AB. $O_3M \perp O_2$ C and

$O_1N \perp O_2C$.

$O_1O_2 = O_1Q + QO_2 = 3+4=7$

$O_2O_3 = O_2P + PO_3 = 4+6=10$

$O_2N = LC - (LO_2+NC) = 12-(4+3) = 5$

$O_2M = LC - (LO_2+MC) = 12 - (4+6) = 2$

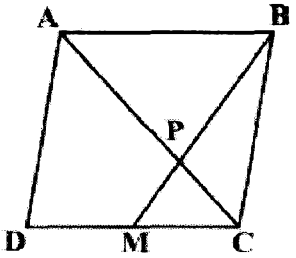
From ΔO_1O_2N, $O_1N = \sqrt{O_1O_2^2 - O_2N^2} = \sqrt{49-25} = \sqrt{24} = 2\sqrt{6}$

From ΔO_2O_3M, $O_3M = \sqrt{O_2O_3^2 - O_2M^2} = \sqrt{100-4} = \sqrt{96} = 4\sqrt{6}$

Now $AB = AC+CB = O_1N + O_3M = 2\sqrt{6} + 4\sqrt{6} = 6\sqrt{6}$

$\therefore a = 6$ Ans. $a = 6$

24. In a parallelogram ABCD, M is the mid point of CD. Also BM meets AC at P. If AP=65 and PM=30, the largest possible integer length of AB is____.



Solu :

Let AB=2x. Then MC=x. $\Delta APB \sim \Delta CPM$ with scale factor 2:1. Since PM = 30, PB = 60. By triangle inequality in Δ APB, we have AP+PB>AB giving 60 + 65 > AB. The largest integer length of AB is 124.

25. The remainder when 1.2.3 + 2.3.4 +.....+97.98.99 is divided by 1000 is_____.

Solu :

The given sum is $3!\sum_{m=3}^{99}\binom{m}{3} = 3!\binom{100}{4}$ by

Hockey stick identity. Since

$$3!\binom{100}{4} = 6\left(\frac{100.99.98.97}{24}\right) = 25.99.98.97 = 23527350$$

the remainder is 350 when divided by 1000.

26. In the addition shown here, each letter represent a digit and different letters represent different digits. The number ELEVEN is___.

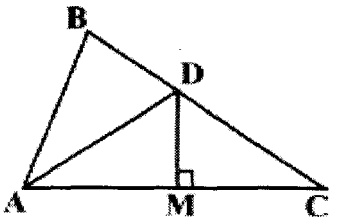
ONE
TWO
TWO
THREE
THREE
ELEVEN

Solu :

The sum of the given numbers is at most $2(98755) + 2(946) + 635 = 200037$. Thus E=1. The one's column shows that N must be odd. Looking at the tens column, we see that the carry comming from the ones column must be even. Since $2O+3E = 2O+3 \leq 21 < 20+N$, this carry must be zero. This gives two possibilities, either O=2 and N=7 or O=3 and N=9. It is easy to see that the first case gives no solution. Consider the second case; from the tens column, we have W=0 or 5. Suppose W=0: From the thnousands column., the carry out of the hundredth column must be odd. Hence it must be 1 or 3. This gives $2R+2T = V+6$ or $2R+2T = V+26$. In the first case, we see that $T \leq 7$ and H=5. From the Ten Thousandth column, we get $2T=L+9$ and $T \geq 5$. Since 5 is not available, we get $T = 6$ and $L = 3 = O$ or $T=7$ and $L=5=H$, neither is possible. In the second case, we see that H=4. Since $R+T \leq 15$. we see that $V \leq 4$ is even. Thus V=2 and $R+T = 14$ and R ,T are 6,8 in some order. But $T = 6$ gives $L=3=O$, and hence we must have T=8, R=6 and L=7. This gives the solution.

391+803+803+84611+84611=171219

27. In the adjoining figure, $\angle B=115^{\circ}$. M is the mid point of side AC. DM $\perp$ AC. $\angle CAD : \angle BAD = 5:3$. If $2\angle A - \angle C = x^{0}$, then x is___.



Solu :

M is mid point of AC and MD $\perp$ AC.

$\therefore \angle CAD = \angle C$.

Given $\angle CAD : \angle BAD = 5:3$, We get $\angle CAD = \frac{5}{8} A = C$

$\therefore 5A = 8C$

Now $A+B+C = 180^0 \Rightarrow A+115 + \frac{5A}{8} = 180^0 \Rightarrow A = 40°$

$\therefore C = 25°$

$\therefore 2\angle A - \angle C = 80° - 25° = 55°$

$\therefore x = 55°$ Ans.

28. There are two piles of cards A and B with cards 1,2,3,4,5,6,7,8 and 1,2,3,4,5 resp. Ria mixes the cards. Starting with pile A and taking cards from the piles and making them into a single pile in order. She never makes more cards from B then A at any instant. For example, the resulting order of the cards may be 1122334455678 or 1234512345678, but not 1223144553678. The number of different orders that can end up is ____

Solu :

					42	132	297	572
				14	42	90	165	275
			5	14	28	48	75	110
		2	5	9	14	20	27	35
	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1

There is a one to one correspondance between possible orderings and the paths of length 13 in the co-ordinate plane

from (0,0) to (8,5) where each step from point (x,y) is one unit to the right to (x+1,y) or one unit up to (x,y+1), and there are no points (x,y) on the path with y>x. Indeed, given a possible ordering, construct a path by reading the numbers in order, and each time a number is seen for the first time, have the path take one step right, R, and each time a number is seen for the second time, have the path take one step up U. From the figure above, it follows that the number of paths is 572.

29. The value of x which satisfies

$x^{-x^{1-x}} = 2^8$ is m. The value of m^2+m-1 is___.

Solu :

$$x^{-x^{1-x}} = 2^8$$

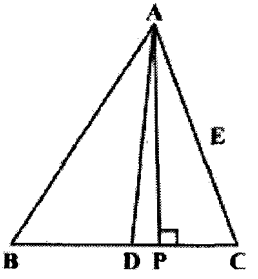
$$x^{-(x \cdot x^{-x})} = 2^{2.4}$$

$$\therefore \left(x^{-x}\right)^{x^{-x}} = \left(2^2\right)^4 = 4^4$$

$\Rightarrow x^{-x}=4 \Rightarrow x^{-x} = 2^2 = (-2)^2 \Rightarrow x = -2$

$\therefore m = -2 \qquad \therefore m^2+m-1 = 4-2-1=1.$

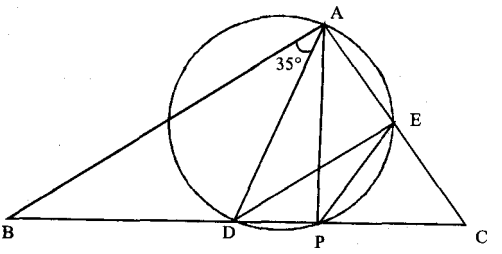
30. In the adjoining figure, D and E are midpoints of BC and AC respectively. AP is the attitude. $\angle BAD = 35^0$. If points D, P, E, A are concyclic, the measure of the angle EPC is____ degrees.



Solu :

D and E are midpoints of BC and AC respectively.

$\therefore$ AB II DE and AD is transversal.



$\therefore \angle ADE = \angle BAD = 35^0$_____ (i)

$\angle$AD E and $\angle$AP E are angles in the same arc $\overset{\frown}{AE}$.

$\therefore \angle APE = \angle ADE = 35^0$____ by (i)..... (ii)

But $\angle APC = 90^0$ given

$\therefore \angle EPC = 90^0 - 35^0 = 55^0$.

NMTC-53 – PRIMARY – MARCH – 2022
FINAL - GAUSS CONTEST
(V & VI STANDARDS)

1. If $A=\frac{1}{18}+\frac{1}{36}+\frac{1}{60}+\frac{1}{90}+\frac{1}{126}+\frac{1}{168}$ and $B=\frac{1}{5}-\frac{5}{30}+\frac{7}{60}-\frac{9}{100}+\frac{11}{150}-\frac{13}{210}$ find the value of $8A + 14B$.

Solution:

$$A=\frac{1}{3}\left[\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}+\frac{1}{56}\right]$$

$$=\frac{1}{3}\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)\right]$$

$$=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{8}\right]=\frac{1}{3}\times\frac{3}{8}=\frac{1}{8}$$

$$B=\frac{1}{5}\left[1-\frac{5}{6}+\frac{7}{12}-\frac{9}{20}+\frac{11}{30}-\frac{13}{42}\right]$$

$$=\frac{1}{5}\left[1-\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)-\left(\frac{1}{4}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{6}\right)-\left(\frac{1}{6}+\frac{1}{7}\right)\right]$$

$$=\frac{1}{5}\left[1-\frac{1}{2}-\frac{1}{3}+\frac{1}{3}+\frac{1}{4}-\frac{1}{4}-\frac{1}{5}+\frac{1}{5}+\frac{1}{6}-\frac{1}{6}-\frac{1}{7}\right]$$

$$=\frac{1}{5}\left[\frac{1}{2}-\frac{1}{7}\right]-\frac{1}{5}\times\frac{5}{14}=\frac{1}{14}$$

$\therefore\ 8A+14B = 1 + 1 = 2.$ Ans: 2

Note: The last term in the value of B in the questions is -ve.

2. *a*) Given n is the smallest positive integer for which $864 \times n$ is a perfect cube. Find n. Three strings of lengths $15m$, $42m$ and

$39m$ are to be cut into pieces of equal lengths. If 'm' is the greatest possible length of each piece, find 'm'.

Find the area of the rectangle whose length and breadth are n metres and m metres respectively.

Solution:

$864 = 8 \times 27 \times 4 = 2^3 \times 3^3 \times 2^2$

$\therefore$ $864 \times n$ to be a perfect square, the smallest n is 2.

$\therefore$ $n=2$.

HCF of 15, 42, 39 is 3

$\therefore$ $M = 3$

$\therefore$ The area of the rectangle = $2 \times 3 = 6$ sq.m.

b) 605 chocolates are distributed equally among some children. The number of chocolates received by each child is 20% of the total number of children. How many chocolates did each child receive?

Solution:

Let number of children = x

as per given $\frac{605}{x} = 20\%$ of $x = \frac{x}{5}$

$\Rightarrow x^2 = 605 \times 5 = 121 \times 5 \times 5 = 11^2 \times 5^2 = (55)^2$

$\therefore$ no. of children =55

$\therefore$ number of chocolates each receives $= \frac{605}{55} = 11$ Ans. 11

Another method:

Let the number of children = x

and number of chocolates each get $= y$

$\therefore\ 605 = y \times x$ and y is 20% $x\ \therefore y = \frac{x}{5}$

$\therefore x = 5y$

$\therefore 605 = 5y^2 \Rightarrow y^2 = 121 \Rightarrow y = 11$ (+ve value) $\therefore$ Ans = 11.

3. John, on his birthday, went to an ice-cream shop along with his five friends. There are three flavours of ice-cream, Butterscotch, Vanilla and Pista. John and his friends do not have any specific choice of a flavour. How many different assortments of the ice-cream can be selected?

Solution:

Along with John there are 6 persons. There are 3 flavours, Butterscotch, Vanilla, Pista. If all of them get (for example) Pista, then the distribution will be 0,0,6. This distribution gives two more options: 6,0,0, 0,6,0.

Similarly 0,3,3 and 1,1,4 will give 3 distributions each.
So total upto now = 9 --- (i)
Now 0,1,5 will give 6 distributions as below:

0,1,5; 0,5,1; 1,0,5; 1,5,0; 5,1,0; 5,0,1 and similarly 0,2,4 and 1,2,3 will give 6 distributions each.
So total = 18 ---- (ii)

Finally: 2,2,2 is the only options = 1 ----- (iii)

$\therefore$ Totally = 9 + 18 + 1 (from (i), (ii) and (iii) = 28

Ans: 28

4. A natural number is divisible both by 9 and 4. This number consists of only 4*s* and 9*s*. Let '*n*' be such a number which is the smallest. Find the last 5 digits of this number.
Solution:

'*n*' is divisible by 4. This means that the last two digits must be divisible by 4. Since the digits have to be 4's and 9's only, the last two digit must be 44.

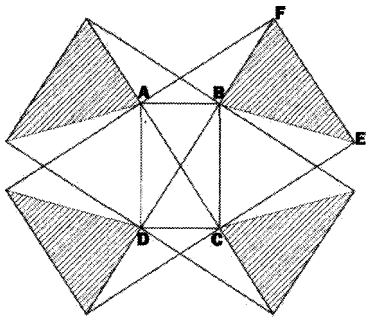
Now '*n*' is divisible by 9. This means that the sum of the digits is divisible by 9. We want the number to be the smallest. Consider 944 or 444. Both are not divisible by 9.

Consider 4944 which is also not divisible by 9. Consider 44944 which is also not divisible by 9. Extending this way, we find the smallest is

4444444944 is divisible by 9.

The last 5 digits are 44944.

5. *a*) In the figure, *ABCD* is a rectangle. Squares are drawn outwardly on the diagonals. (For example, on *AC*, the square drawn is *ACEF*).
The dimensions of the rectangle are 5 *cm* × 20 *cm*.
Find the total area of the shaded region.



Solution: (a)

Draw lines through *B* parallel to the side of the square. These two lines form four rectangles with corresponding diagonals. The letters *a,b,c,d* denote the area of the equal triangles.

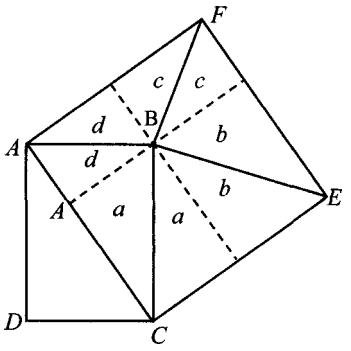
Clearly $2(a+b+c+d)$ = area of the sq. $ACEF = 425$. – – (1)

Consider the triangle ABC.

Area of the ΔABC

$= \frac{1}{2}$ (area of the triangle ABC) $= a + d$

$\therefore 50 = a + d$



(i) $\Rightarrow 2(b + c) + 2(a + d) = 425$

$\Rightarrow 2(b + c) = 325$

Area of the

$\Delta FBE = b + c = \frac{325}{2}$ cm^2.

$\therefore$ Total shaded areas

$= 4 \times \frac{325}{2}$ cm^2 $= 650$ cm^2

Alternate method:

Area of $\Delta ABC = \frac{1}{2}\sqrt{425}.BP = 50$

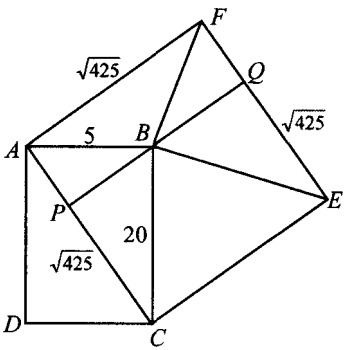
$\Rightarrow BP = \frac{100}{\sqrt{425}}$

$\Rightarrow BQ = \sqrt{425} - \frac{100}{\sqrt{425}}$

Area of

$\Delta BFE = \frac{1}{2}\sqrt{425}\left(\sqrt{425} - \frac{100}{\sqrt{425}}\right)$

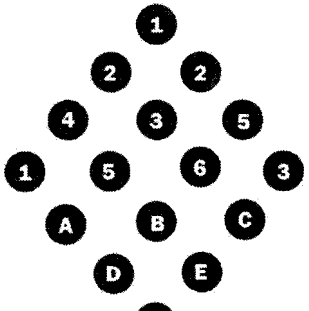
$= \frac{1}{2}(325)$



$\therefore$ Total area of 4 regions

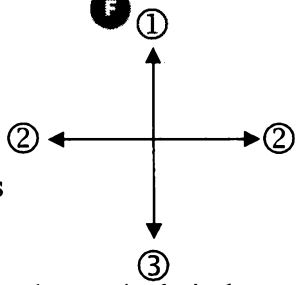
$= 4 \times \frac{1}{2} \times 325 = 650\,\text{cm}^2$

b) From the pattern shown in the adjoining figure, find the values of A, B, C, D, E and F.
Also compute the value of
$A^2+C^2+F^2-B-D-E$.



Solution:

Consider the four circles at the top.
We find that 1+3 = 2 + 2.
Thus the horizontal sum of the numbers



= The sum of the numbers in the vertical circles.

Observe, this is happening in the upper part of the figure.

Thus

$A+4=1+5=6 \Rightarrow A=2$
$B+3=5+6=11 \Rightarrow B=8$
$C+5=6+3=9 \Rightarrow C=4$
$D+5=2+8=10 \Rightarrow D=5$
$E+6=8+4=12 \Rightarrow E=6$
$F+8=5+6=11 \Rightarrow F=3$

Now

$A^2+C^2+F^2-B-D-E=4+16+9-8-5-6=10$ Ans.10

NMTC-53 – SUB-JUNIOR – MARCH – 2022
FINAL - KAPREKAR CONTEST
(VII & VIII STANDARDS)

1. *a*) In the grid given here, the numbers against each row or column is the sum of all the numbers in that respective row or column.
For example, in the top row,
$a + b + c + d = 17$, etc.

a	*b*	*c*	*d*	**17**
a	*c*	*c*	*b*	**19**
c	*b*	*a*	*b*	**15**
d	*d*	*a*	*b*	**15**
16	**18**	**18**	**14**	

Find the values of *a*, *b*, c and *d*.

Solution :

There are 8 equations:

a + b + c + d = 17 ---------- (1)
a + b + 2c = 19 ---------- (2)
a + 2b + c = 15 ---------- (3)
a + b + 2d = 15 ---------- (4)
2a + c + d = 16 ---------- (5)
2b + c + d = 18 ---------- (6)
2a + 2c = 18 ---------- (7)
3b + d = 14 ---------- (8)

First method :

(7) $\Rightarrow a + c = 9$---------- (9)

putting in (1) gives

$b + d = 8$ ---------- (10)

(8)$\Rightarrow$ $3b + d = 14$ using (10), $2b = 6$ $\Rightarrow b = 3$

$(10) \Rightarrow \quad 3 + d = 8 \qquad \Rightarrow d = 5$

$(4) \Rightarrow \quad a + b + 2d = 15 \Rightarrow a + 3 + 10 = 15 \qquad \Rightarrow a = 2$

$(1) \Rightarrow \quad a + b + c + d = 17$

$\Rightarrow \quad 2 + 3 + c + 5 = 17 \qquad \Rightarrow c = 7$

Ans: $\therefore a = 2, b = 3, c = 7, d = 5.$

Second method :

$(7) \Rightarrow a + c = 9$

$(3) \Rightarrow a + c + 2b = 15 \Rightarrow 2b = 6 \Rightarrow b = 3$

$(2) \Rightarrow a + 2c = 16$

$a + c = 9 \Rightarrow c = 7$

$(7) \Rightarrow a + c = 9 \Rightarrow a = 2$

$(8) \Rightarrow 3b + d = 14 \Rightarrow 9 + d = 14 \Rightarrow d = 5$

$\therefore$ Answer $a=2, b= 3, c = 7, d = 5$

Note:

This problem can be solved by several different methods. Students are advised to observe the combinations of a,b,c & d in the equations and then develop a strategy that can give the values in fewer steps.

b) If $\frac{a}{b}+\frac{b}{a} = x, \quad \frac{b}{c}+\frac{c}{b} = y, \quad \frac{c}{a}+\frac{a}{c} = z$ find the numerical value of $(x^2 + y^2 + z^2 - xyz)$.

Solution:

$$x^2 + y^2 + z^2 = \left(\frac{a}{b}+\frac{b}{a}\right)^2 + \left(\frac{b}{c}+\frac{c}{b}\right)^2 + \left(\frac{c}{a}+\frac{a}{c}\right)^2$$

$$= \frac{a^2}{b^2}+\frac{b^2}{a^2}+2+\frac{b^2}{c^2}+\frac{c^2}{b^2}+2+\left(\frac{c}{a}+\frac{a}{c}\right)^2$$

$$= \frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{b^2}{c^2} + \frac{c^2}{b^2} + \left(\frac{c}{a} + \frac{a}{c}\right)^2 + 4$$

$$= b^2\left(\frac{1}{a^2} + \frac{1}{c^2}\right) + \frac{1}{b^2}\left(a^2 + c^2\right) + \left(\frac{c}{a} + \frac{a}{c}\right)^2 + 4$$

$$= \frac{b^2}{ac}\left(\frac{a}{c} + \frac{c}{a}\right) + \frac{ac}{b^2}\left(\frac{a}{c} + \frac{c}{a}\right) + \left(\frac{c}{a} + \frac{a}{c}\right)^2 + 4$$

$$= \left(\frac{a}{c} + \frac{c}{a}\right)\left(\frac{b^2}{ac} + \frac{ac}{b^2} + \frac{c}{a} + \frac{a}{c}\right) + 4$$

$$= \left(\frac{a}{c} + \frac{c}{a}\right)\left(\frac{b^2}{ac} + \frac{a}{c} + \frac{ac}{b^2} + \frac{c}{a}\right) + 4$$

$$= \left(\frac{a}{c} + \frac{c}{a}\right)\left[\frac{b}{c}\left(\frac{b}{a} + \frac{a}{b}\right) + \frac{c}{b}\left(\frac{a}{b} + \frac{b}{a}\right)\right] + 4$$

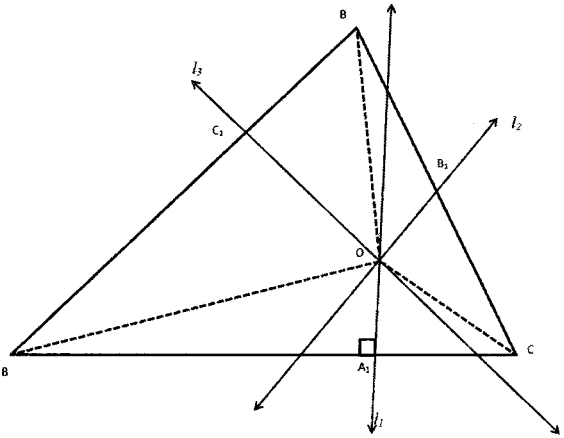
$$= \left(\frac{a}{c} + \frac{c}{a}\right)\left(\frac{a}{b} + \frac{b}{a}\right)\left(\frac{b}{c} + \frac{c}{b}\right) + 4$$

$\therefore \quad x^2 + y^2 + z^2 = xyz + 4$

$\therefore \quad x^2 + y^2 + z^2 - xyz = 4$ Ans. 4.

2. ABC is a triangle. l_1, l_2, l_3 are three lines that are perpendicular to the sides BC, CA and AB at A_1, B_1 and C_1 respectively. Prove that l_1, l_2, l_3 are concurrent if and only if $A_1B^2 + C_1A^2 + B_1C^2 = A_1C^2 + C_1B^2 + B_1A^2$

Solution :



(A) part 1

Let l_1, l_2, l_3 be concurrent. Using Pythagoras theorem, we get,

$A_1B^2 = BO^2 - OA_1^2$

$C_1A^2 = AO^2 - OC_1^2$

$B_1C^2 = CO^2 - OB_1^2$

$\Rightarrow A_1B^2+C_1A^2+B_1C^2=AO^2+BO^2+CO^2 - OA_1^2 - OB_1^2 - OC_1^2$ --- (1)

Similarly,

$A_1C^2 = OC^2 - OA_1^2$, $C_1B^2 = OB^2 - OC_1^2$, $B_1A^2 = OA^2 - OB_1^2$

Adding, we get,

$A_1C^2+C_1B^2+B_1A^2 = OA^2+OB^2+OC^2-OA_1^2-OB_1^2-OC_1^2$ ---(2)

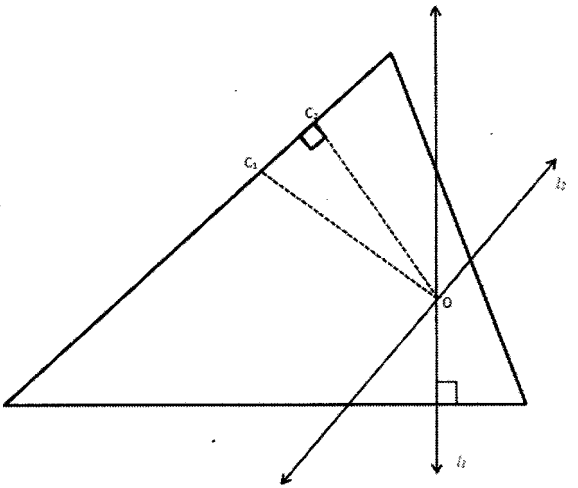
Equating (1) and (2) we get

$A_1B^2 + B_1C^2 + C_1A^2 = AB_1^2 + BC_1^2 + CA_1^2$

(B) part 2

Let the l_1, l_2, l_3 which are perpendicular to the respective sides BC, CA, AB such that

$A_1B^2 + B_1C^2 + C_1A^2 = AB_1^2 + BC_1^2 + CA_1^2$ (given) --- (*)



To proud that l_1, l_2, l_3 are concurrent
Let l_1, l_2 intersect at '0'

Let OC_2 be the perpendicular from O to AB.

Using the above result we have
$A_1B^2 + B_1C^2 + C_2A^2 = A_1C^2 + C_2B^2 + B_1A^2$

Using (*) we get

$$C_2A^2 - C_1A^2 = C_2B^2 - C_1B^2$$

$$\Rightarrow C_2A^2 - C_2B^2 = C_1A^2 - C_1B^2$$

$$\Rightarrow (C_2A + C_2B)\ (C_2A - C_2B) = (C_1A + C_1B)\ (C_1A - C_1B)$$

$\Rightarrow AB\,(C_2A - C_2B) = AB\,(C_1A - C_1B)$
$\Rightarrow C_2A - C_2B = C_1A - C_1B$

$\Rightarrow C_1, C_2$ coinside

$\therefore l_3$ passes through O, $\therefore$ l_1, l_2, l_3 are concurrent.

3. *a*) If a,b are real numbers each greater than 1. Prove:
$$(1+a)(1+b) < 2(ab + 1).$$

Use this result to prove
$$8(abcd + 1) > (a + 1)\,(b + 1)\,(c + 1)\,(d + 1),$$

where a, b, c, d are real numbers each greater than 1.

Solution :

$a>1$, $b>1$
Let $S = (1 + a)\,(1 + b) - 2(ab + 1)$
$= 1 + a + b + ab - 2ab - 2$
$= a + b - ab - 1$
$= (a - 1)\,(1 - b)$

Now $a>1 \Rightarrow a - 1> 0$ and $b>1 \Rightarrow 1-b< 0$

$\therefore S<0$

$\therefore (1+a)\,(1+b) - 2\,(ab +1) < 0$
$\Rightarrow (1+a)\,(1+b) < 2\,(ab+1)$ .... (1)
i.e., $2(ab+1) > (1+a)(1+b)$ .... (*)

Now, $c>1$, $d>1$
$\Rightarrow 2(cd+1) > (1+c)\,(1+d)$

Using (*)

$4(ab+1)(cd+1) > (1+a)(1+b)(1+c)(1+d)$

$\Rightarrow (1+a)(1+b)(1+c)(1+d) < 4(ab+1)(cd+1)$ ....(2)

Now,

$a>1,\ b>1 \Rightarrow ab>1$ and

$c>1,\ d>1 \Rightarrow cd>1$

Using (1), we get

$$(1+ab)(1+cd) < 2(1+abcd)$$

$$\Rightarrow 2(1+abcd) > (1+ab)(1+cd)$$

$\therefore$ (2) $\Rightarrow (1+a)(1+b)(1+c)(1+d) < 8(abcd+1)$

b) The seven digit number 1234 ✶ ✶ ✶ is a perfect square number, where each ✶ represents a digit (may be same or distinct). Find the square root of this seven digit number.

Solution:

The number lies between 1234000 and 1234999.

Now $\sqrt{1234000} \approx 1110.85$ and

$\sqrt{1234999} \approx 1111.30$

The square root of the given numbers lies between these two.

Clearly it is 1111.

and hence the given number is 1234321.

4. *a*) If *a, b, c are* integers (*a, b, c* are not zero) such that $a + b + c = 0$, prove that $(2a^4 + 2b^4 + 2c^4)$ is the square of an integer.

Solution:

$$\begin{aligned}2a^4 + 2b^4 + 2c^4 &= 2[(a^2)^2 + (b^2)^2 + (c^2)^2] \\ &= 2[(a^2 + b^2)^2 - 2a^2b^2 + c^4] \\ &= 2[(c^2 - 2ab)^2 - 2a^2b^2 + c^4] \\ &= 2[(c^4 - 4c^2ab - 4a^2b^2 - 2a^2b^2 + c^4] \\ &= 2[2c^4 - 4c^2ab + 2a^2b^2] \\ &= 4(c^4 - 2c^2ab + a^2b^2) \\ &= 4(c^2 - ab)^2 \\ &= [2(c^2 - ab)]^2 \text{ which is the square of an integer.}\end{aligned}$$

a) Find the smallest positive integer 'n' such that $4^{27} + 4^{1000} + 4^n$ is a perfect square.

Solution:

$4^{27} + 4^{1000} + 4^n = 4^{27}\,[1 + 4^{973} + 4^{n-27}]$

$4^{973} = (2^2)^{973} = (2^{973})^2 < 1 + 4^{973} + 4^{n-27}$

Thus $1 + 4^{973} + 4^{n-27}$ to be a perfect square, has to be at least $(2^{973} + 1)^2$.

$1 + 4^{973} + 4^{n-27} = (4^{973})^2 + 2.2^{973} + 1$

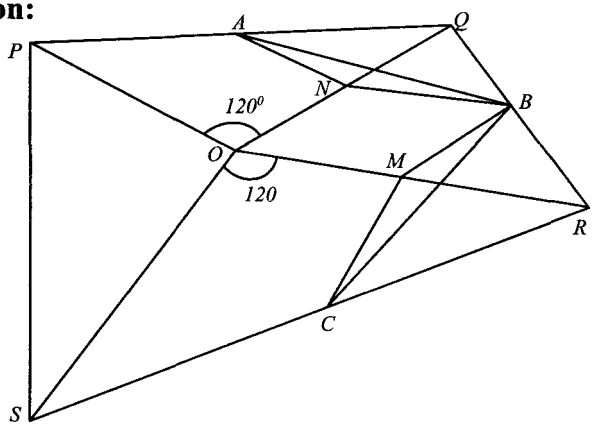
$\Rightarrow\ 2^{2(n-27)} = 2^{974} \Rightarrow\ 2(n-27) = 974$

$\Rightarrow n = 514.$

This is the least value required.

5. $PQRS$ is a convex quadrilateral. Point O is inside the quadrilateral such that $OP = OQ$ and $OS = OR$. If $\angle POQ = \angle SOR = 120^O$ and A, B, C are respectively the mid points of PQ, QR and RS, prove that ABC is an equilateral triangle.

Solution:



Let N, M be respectively the midpoint of OQ and OR.

$NB||OR$ and $NB = \frac{1}{2} OR$

$= \frac{1}{2} OS = MC$. i.e. $NB = MC$ ---(1)

$AN||OP$ and $AN = \frac{1}{2} OP$

$= \frac{1}{2} OQ = MB$ i.e. $AN = MB$ ---(2)

Since NB and NA are parallel to OR and OP.

$\angle ANB = 120^0 + \angle QOR$ and

$\angle BMC = 120^0 + \angle QOR$ $\quad\quad\quad\quad$ $\therefore \angle ANB = \angle BMC$ ---(3)

From (1), (2) and (3), $\Delta ANB \cong \Delta BMC$

$\therefore AB = BC$ ---(4)

Let

$\angle ABN = x$ then $\angle BCM = x$ and

$\angle BAN = y$ then $\angle CBM = y$ and

$\angle NBM = z$ then $\angle BMR = z$

Now $\angle ABC = x + z + y$

$= (x + y + z + 120^0) - 120^0$ ($\because$ angles of $\Delta\, BMC$)

$= 180^0 - 120^0$

$\therefore \angle ABC = 60^0$ ---(5)

From (4) and (5), ΔABC is equilateral triangle.

NMTC-53 - JUNIOR-2022
FINAL TEST - BHASKARA CONTEST
(Std IX & X)

1. a) If $a^2y^4 + b^2x^4 = a^2b^2$, $a^2+b^2 = x^2+y^2 = 1$, Prove that

$a^4y^6+b^4x^6 = (a^2y^4+b^2x^4)^2$

Solu :

Given $a^2y^4+b^2x^4 = a^2b^2$ & $x^2+y^2=1$,

$a^2y^4+b^2x^4=a^2b^2\,(x^2+y^2)$

$\Rightarrow$ $a^2y^4-a^2b^2y^2 = a^2b^2x^2 - b^2x^4$

$\Rightarrow$ $a^2y^2\,(y^2-b^2) = b^2x^2\,(a^2-x^2)$

$\Rightarrow$ $b^2x^2-a^2y^2 = 0$ $(\because\ a^2+b^2=x^2+y^2 \Rightarrow a^2-x^2=y^2-b^2)$

$\Rightarrow$ $(b^2x^2-a^2y^2)^2 = 0$

$\Rightarrow$ $b^4x^4+a^4y^4 - 2\ a^2b^2x^2y^2 = 0$

$\Rightarrow$ $b^4x^4+a^4y^4 = 2a^2b^2x^2y^2$ _____(i)

Now Take,

$$\begin{aligned} a^4y^6+b^4x^6 &= (a^4y^6+b^4x^6)\,(x^2+y^2) \\ &= a^4x^2y^6 + a^4y^8 + b^4x^8 + b^4x^6y^2 \\ &= x^2y^2\,(a^4y^4+b^4x^4) + b^4x^8 + a^4y^8 \\ &= x^2y^2\,(2a^2b^2x^2y^2) + b^4x^8 + a^4y^8 \text{ (by (i)} \\ &= (a^2y^4 + b^2x^4)^2. \end{aligned}$$

(b) Determine all integers m, n such that
$(m^3 + n)\,(n^3 + m) = (m + n)^4$

Solv : Clearly the trivial solutions are (o, k) and (k,o), $K \in Z$

The given equation reduces to

$m^3n^3 + mm = 4m^3n + 6m^2n^2 + 4mn^3$ ___(i) [Using the expansion of $(a+b)^4$]

When mn = o, either m = o or n=o and we get the trivial solutions.

Let $mn \neq o$ then dividing (i) by mn,

$m^2n^2 + 1 = 4m^2 + 6mn + 4n^2$

Adding 2mn on both sides, we get
$(mn+1)^2 = (2m + 2n)^2$

$\Rightarrow (mn+1)^2 - (2m+2n)^2 = 0$

$\Rightarrow mn + 1 + 2m + 2n = 0$ or $mn + 1 - 2m - 2n = 0$

$\Rightarrow (m+2)(n+2) = 3$ - (ii) or $(m-2)(n-2) = 3$ __ (iii)

(ii) $\Rightarrow$ $m+2 = 1, n+2=3$

$\Rightarrow m=-1, n=1$ $\therefore (-1, 1)$

or

$m+2=3, n+2=1$

$\Rightarrow m=1, n-1$ $\therefore (1,-1)$

or

$m+2=-1, n+2= -3$

$\Rightarrow m=-3, n=-5$ $\therefore (-3, -5)$

or

$m+2 = -3, n-2= -1$

$\Rightarrow m= -5, n= -3$ $\therefore (-5, -3)$

(ii) $\Rightarrow$ $m-2=1, n-2=3$

$\Rightarrow m-3, n=5$ $\therefore (3, 5)$

or

$m-2=3, n-2=1$

$\Rightarrow m=5, n=3$ $\therefore (5, 3)$

or

$m-2 = -1,\ n-2=-3$

$\Rightarrow$ $m=1,\ n= =1$ $\quad\therefore (1,-1)$

or

$m-2= -3,\ n-2= -1$

$\Rightarrow$ $m-1,\ n-1$ $\quad\therefore (-1, 1)$

Thus the distinct solutions are :

$(o, k), (k, o)$; $K \in Z$ and

$(-1,1)\ (1,-1), (3,5)\ (5,3), (-3,-5), (-5, -3)$

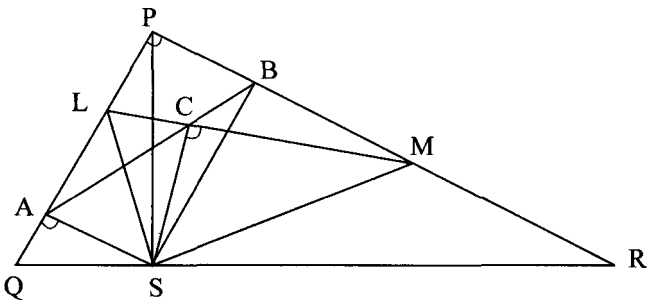
2. In ΔPQR, $P=90°$, PS is the altitude. From S drawn SA, SB respectively perpendiculars to PQ and PR, C is an arbitrary point on the line segment AB. The line perpendicular to SC meet PQ at L and PR at M.

Prove : (i) The circumcricle of PLM passes through S.

(ii) If C_1 is another arbitrary point on the line segment AB and the line perpendicular to SC_1, meet PQ at L_1 and PR at M_1 prove that

$\frac{LL_1}{MM_1}$ is a constant.

Solu :



(i) Cleraly SCBM is a cyclic quadrilateral. Similarly SCLA is also a cyclic quadrilateral.

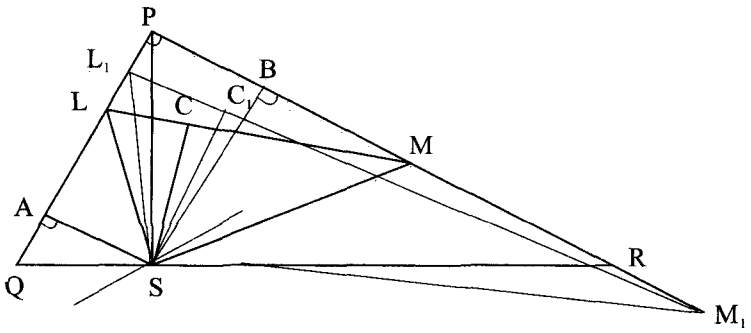
$\angle ALS = \angle ACS = 180^0 - \angle SCB = \angle SMB$

$\therefore \angle ALS = \angle SMB$

$\Rightarrow$ SMPL is cyclic quadrilateral.

Thus, the circumericle of PLM passes through S

ii) Take another point C_1 on AB, Join SC_1. Draw the line $\perp$ to SC_1 which meets PQ at L_1 and PR produced at M_1



To prove : $\frac{LL_1}{MM_1}$ is constant.

Consider the similar triangles SAL and SBM,

We have $\frac{AL}{BM} = \frac{SA}{SB}$ _____(1)

SImilarly for similar triangles SAL_1 and SBM_1, we have

$$\frac{AL_1}{BM_1} = \frac{SA}{SB} \quad \text{_____(2)}$$

$$\therefore \frac{AL_1}{BM_1} = \frac{AL}{BM} = \frac{SA}{SB}$$

$$\Rightarrow \frac{SA}{SB} = \frac{AL_1\text{-}AL}{BM_1\text{-}BM} = \frac{LL_1}{MM_1}$$

Since $\frac{SA}{SB}$ is constant, $\frac{LL_1}{MM_1}$ is also constant

3. (a) Two concentric discs each divided into 100 sections, as in the diagram. 50 sectors on each disc are selected at random and painted. The centre disc is rotated to any arbitrary position and the number of coloured regions match are counted. Show that regardless of the initial colouring there must be some position that has at least 50 matches.

   b) Three natural numbers form an increasing Geometric progression whose sum is 57. Detemine the progression.

Solu :

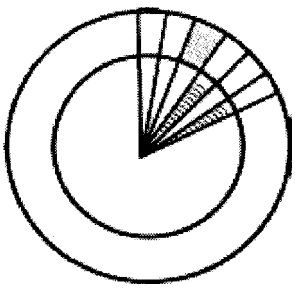
a) Look at one particular sector as the inner disc moves through all possible positions. It will match with an outer sector exactly 50 times. Since this is true for each inner sector, there is a total of $100 \times 50 = 5000$ matches for the 100 positions. Thus the average number of matches per position is 50. So there must be at least one position that has 50 or more matches.

b) Let the common ratio be $\frac{p}{q}$ where

$(p, q) = 1$.

Let the numbers be $\frac{qa}{p}$, a, $\frac{ap}{q}$

$q|a, p|a \quad \therefore pq|a \quad \therefore a = kpq$



$\therefore$ The G.P. is kq^2, kpq, kp^2

We need the positive integer solution to

$k(p^2 + pq + q^2) = 57$; $(p, q) = 1$, $p > q$

So the second factor is at least 7 and divides 57 (=3x19)

$\therefore$ $k = 1$, $p^2 + pq + q^2 = 57$ or $k=3$, $p^2 + pq + q^2 = 19$

(i) $p^2+pq+q^2 = 19$

When $q=1$, $p^2+p=18 \Rightarrow (2p+1)^2 = 73$ (not a square)

$q \geq 2$, Then $p \geq 3$; $p^2+pq+q^2 \geq 3^2 + 2.3 + 2^2 = 19$

Equality holds when $p = 3$, $q = 2$

$\therefore$ The progression is (12, 18, 27)

(ii) $p^2 + pq + q^2 = 57$, $q \leq 3$ since otherwise

$p^2 + pq + q^2 \geq 5^2+4.5+4^2 = 61 > 57$

If $q=1$, $p^2+p=56$, $(2p+1)^2 = 225 \Rightarrow 2p+1=15$

$\Rightarrow p = 7$

$\therefore$ The G.P. is (1,7,49)

(iii) $q=2$, $p^2+2p = 53$, $(p+1)^2 = 54$; not a square

$q=3$, $p^2+3p = 48$, $(2p+3)^2 = 201$; not a square.

So finally we have two propressions (1,7,49) and (12,18, 27)

4. (a) Solve the equation for x :

$(a+b+x)^3 - 4(a^3+b^3+x^3) = 12abx$

(b) If a, b, c are positive integers, prove :

$$a^{\frac{a}{a+b+c}} \cdot b^{\frac{b}{a+b+c}} \cdot c^{\frac{c}{a+b+c}} \geq \frac{a+b+c}{3}$$

Solu : (a) We have

$(a+b+x)^3 = a^3 + b^3 + x^3 = 3a^2(b+x) + 3b^2(a+x) + 3x^2(a+b) + 6abx$

Using this the equation reduces to

$x^3-(a+b)x^2 - (a-b)^2 x + (a-b)^2(a+b) = 0$

$\Rightarrow x^2(x - a - b) - (a - b)^2(x - a - b) = 0$

$\Rightarrow$ $(x-a-b)[x^2-(a-b)^2]=0$

$\Rightarrow$ $x=a+b,\ x=a-b,\ x=b-a$

(b) Since a, b, c are positive integers,

Take $\frac{1}{a}, \frac{1}{a}, \frac{1}{a}$ .....'a' times

$\frac{1}{b}, \frac{1}{b}, \frac{1}{b}$ .....'b' times

$\frac{1}{c}, \frac{1}{c}, \frac{1}{c}$ .....'c' times

Now $a.\frac{1}{a}+b.\frac{1}{b}=c.\frac{1}{c}=3$

$\therefore$ G.M. $= \sqrt[a+b+c]{\frac{1}{a^a}.\frac{1}{b^b}.\frac{1}{c^c}}$ and

$$\text{A.M.} = \frac{a.\frac{1}{a}+b.\frac{1}{b}+c.\frac{1}{c}}{a+b+c} = \frac{3}{a+b+c}$$

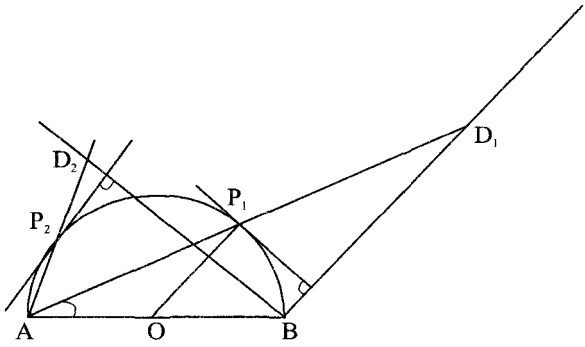
$$\therefore \frac{3}{a+b+c} \geq \sqrt[a+b+c]{\frac{1}{a^a}.\frac{1}{b^b}.\frac{1}{c^c}}$$

$$\therefore \frac{3}{a+b+c} \geq a^{\frac{1}{a/a+b+c}}.b^{\frac{1}{b/a+b+c}}.c^{\frac{1}{c/a+b+c}}$$

$$\therefore a^{\frac{a}{a+b+c}}.b^{\frac{b}{a+b+c}}.c^{\frac{c}{a+b+c}} \geq \frac{a+b+c}{3}$$

5. AOB is a diameter of a semicircle with O as the centre. P is a variable point on the semicircle. The perpendicular from B to the tangent at P and the line AP extended meet at D. Find the locus of D.

Solu :



Take point P_1 on the semicircle. Let the perpendicular from B to the Tangent at P_1 and AP_1 extended meet at D_1. Similarly take another point P_2 and get D_2 for it.

OP_1 II BD_1 as both are $\perp$ to the same line.

$\therefore \angle BD_1P_1 = \angle OP_1A = \angle OAP_1$

$\therefore$ In ABD_1, $AB = BD_1$

Similarly $AB = BD_2$ $\therefore$ $BD_1 = BD_2$

$\therefore$ The locus of D is a semi circle with centre B and radius AB.

NMTC-53 - INTER -2022
FINAL TEST - RAMANUJAN CONTEST
(Std XI & XII)

1. Find the number of divisors of 6^{2022} whose units digit is 6.

Solu : $6^{2020} = 2^{2022} \times 3^{2022}$

Any divisor of 6^{2022} is of the form $2^m x 3^n$ Clearly 2^m ends with 2,4,8 or 6 according as m=1,2,3,0 (mod 4) and 3^n ends with 3,9,7,1 according as n=1,2,3,0 (mod 4) $\therefore$ $2^m x 3^n$ ends with digit 6 if and only if, m,n leave the same remainder when divided by 4.

When m, n are both divisible by 4, the possible values are 0,4,...., 2020. Clearly m > 0 $\therefore$ possible values of m are 4,8,....,2020 and possible values of n are, 0,4,.....2020. Hence the number of possible values are 505 x 506.

Similarly, when m, n both leave remainder 1, the number of possible values of m, n are 1,5,9,...., 2021. There are $(506)^2$ values. When they leave a remainder 2, the possible values of m, n are 2,6,....,2022 i.e. $(506)^2$. For remainder 3, the number of possible values are 3,7,11,...,2019 i.e. $(505)^2$.

$\therefore$ Total number of factors that end with 6 is 505 x 506 + $(506)^2 + (506)^2 + (505)^2$.

2. Do there exist integer pairs (a,b) such that $a^{a+b} = b^{12}$ and $b^{a+b} = a^3$? If so find all, if not, why?

Solu : Dividing the equations, we get,

$$\left(\frac{a}{b}\right)^{a+b} = \left(\frac{b}{a}\right)^3 b^9 \Rightarrow \left(\frac{a}{b}\right)^{a+b+3} = b^9 \qquad - (1)$$

This means b divides a.

Let a=bk where k is an integer. Substituting and simplifying in (1)

$K^{(K+1)b+3} = b^9$

of p is a prime factor of b, p cannot be more than 2, by noting the power (k+1) b+3. So clearly b=2, k=2 is a solution and there are no others.

Thus, (a,b) = (1,1), (1,-1), (4,2)

3. Suppose that x,y, z are real numbers such that

$$x+y+z = 3$$
$$x^3+y^3+z^3 = 15$$
$$x^4+y^4+z^4 = 35$$

and $x^2+y^2+z^2 < 10$, find the value of $x^5+y^5+Z^5$.

Solu : Let $x^2 + y^2 + z^2 = t$. Then

$$xy + yz + zx = \frac{(x+y+z)^2 - (x^2+y^2+z^2)}{2} = \frac{9-t}{2}$$

Also,

$3xyz = x^3+y^3+z^3 - (x+y+z)(x^2+y^2+z^2-xy-yz-zx) =$

$$15 - 3\left(t - \frac{9-t}{2}\right)$$

and hence $xyz = 3\left(\frac{9-t}{2}\right) - 4.$

Now $35 = x^4+y^4+z^4$

$= (x^3+y^3+z^3)(x+y+z) - (x^2+y^2+z^2)(xy+yz+zx) + xyz(x+y+z)$

$$= 45 - \frac{t(9-t)}{2} + \frac{9(9-t)}{2} - 12$$

This yields the quadratic $t^2 - 18t + 77 = 0$ and hence $t = 7, 11$
Since $x^2+y^2+z^2 < 10$. we have $t = 7$. This gives

$$xyz = -1,\ xy+yz+zx = 1$$

and x, y, z are solutions of the cubic $y^3-3y^2+y+1=0$.

$$x^5+y^5+z^5 = (x^4+y^4+z^4)(x+y+z)$$
$$-(x^3+y^3+z^3)(xy+yz+zx) + xyz(x^2+y^2+z^2)$$
$$= 35.3 - 15.1 + 7.(-1) = 83$$

4. If a_1, a_2, a_3, a_4 are positive real numbers, show that

$$\frac{a_1+a_3}{a_1+a_2} + \frac{a_2+a_4}{a_2+a_3} + \frac{a_3+a_1}{a_3+a_4} + \frac{a_4+a_2}{a_4+a_1} \geq 4$$

Solu : If a, b are non negative real numbers, we have $(a+b)^2 \geq 4ab$. Thus

$$(a_1+a_2+a_3+a_4)^2 \geq 4(a_1+a_3)(a_2+a_4)$$

$$\Rightarrow \quad \frac{1}{(a_1+a_3)(a_2+a_4)} \geq \frac{4}{(a_1+a_2+a_3+a_4)^2}$$

Similarly,

$$\frac{1}{(a_1+a_4)(a_2+a_3)} \geq \frac{4}{(a_1+a_2+a_3+a_4)^2}$$

Now,

$$\frac{a_1+a_3}{a_1+a_2} + \frac{a_2+a_4}{a_2+a_3} + \frac{a_3+a_1}{a_3+a_4} + \frac{a_4+a_2}{a_4+a_1}$$
$$= \frac{(a_1+a_3)(a_1+a_2+a_3+a_4)}{(a_1+a_2)(a_3+a_4)} + \frac{(a_2+a_4)(a_1+a_2+a_3+a_4)}{(a_1+a_2)(a_3+a_4)}$$

$$= \frac{4\,(a_1+a_3)}{a_1+a_2+a_3+a_4} + \frac{4(a_2+a_4)}{(a_1+a_2+a_3+a_4)}$$

$$= 4$$

5. The vertices of a regular hexagon drawn on a wall has to be painted with 5 colors such that no two adjacent vertices have the same color. In how many ways can this be done?

We give two solutions.

Solution 1

Without loss of generally, let us assume that the top vertex has color C_1. The following table gives the number of possible colorings of the other vertices (in each row *i*, the number under coloumn C_j donotes that number of valid colorings with the vertex *i* colored with color C_j.

n	C_1	C_2	C_3	C_4	C_5
1	1	0	0	0	0
2	0	1	1	1	1
3	4	3	3	3	3
4	12	13	13	13	13
5	52	51	51	51	51
6	0	205	205	205	205

The required number of colorings is

$$5 \times (205+205+205+205) = 4100$$

Solution 2

Let $f(n)$ be the number of valid ways to color n vertices, so the answer is given by f (6).

For n = 2, we compute $f(n) = 5.4=20$. For $n \geq 3$, we can count the number of valid colorings as follows; choose one of the sections arbitrarily, which we may color 5 ways. Moving clockwise around the ring, there are 4^{n-1} ways to color each of the (n-1) other sections. Therefore, we have 5.4^{n-1} colorings of *n* vertices.

However, note that the first and last vertices could be the same color under this method. To count these invalid colorings, we see that by "merging" the first and last vertices into one, we get a valid coloring of a (n-1) - vertices. That is, there are f (n-1) colorings of an n vertices in which the first and last sections have the same color. Thus, $f(n) = 5.4^{n-1} - f(n-1)$ for all $n \geq 3$.

To compute the requested value f (6), we repeatedly apply this formula :

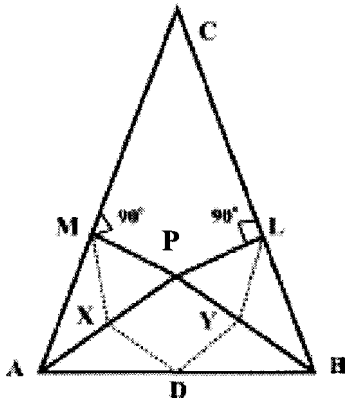
$$f(6) = 5.4^5 - f(5)$$
$$f(5) = 5.4^4 - f(4)$$
$$f(4) = 5.4^3 - f(3)$$
$$f(3) = 5.4^2 - f(2)$$
$$f(2) = 20$$

Thus

$$f(3) = 60, \quad f(4)=260, \quad f(5) = 1020, \quad f(6) = 4100$$

6. Let ÀBC be triangle, and let P be a point inside it such that $\angle PAC = \angle PBC$. The perpendiculars from P to BC and CA meet these lines at L and M respectively, and D is the mid point of AB. Show that DL=DM.

Solu : Let X and Y be the mid points of the segments AP and BP respectively. DYPX is a parallelogram. Since X and Y are circumcenters of the triangles APM and BPL, we conclude that XM = XP = DY and YL=YP = DX. Furthermore,



we have $\angle DXM = \angle DXP + \angle PXM = \angle DXP + 2\angle PAM$. Similarly, $\angle DYL = \angle DYP + \angle PYL = \angle DYP + 2\angle PBL$. Thus $\angle DXM = \angle DYL$. Hence the triangles DXM and LYD are congruent, implying DM = DL.

7. For a positive integer n not divisible by 211, let $f(n)$ denote the smallest positive integer k such that $n^k - 1$ is divisible by 211. Find the reminder when $\sum_{n=1}^{210} nf(n)$ is divided by 211.

Solu : 211 is a prime and 210 = 2.3.5.7. By Fermat's theorem, for any n, $1 \le n \le 210$, we have $n^{210} \equiv 1 \pmod{211}$.

Claim : $f(n) \mid 210$: Suppose that $f(n) = k$. Then k is the smallest positive integer such that $n^k \equiv 1 \pmod{211}$. If $k \mid 210$, then $210 = mk + l$, where $0 < l < k$. Now,

$1=n^{210} \pmod{211} = \left(n^k\right)^{m-n'} \pmod{211} = n^{l} \pmod{211}$

Since $0 < l < k$, this is contradiction.

Let us call $f(n)$, the order of n. Note that the desired sum can be written as

$$\sum_{K|210} k(sum\ of\ numbers\ of\ order\ k)$$

We discuss the different cases based on the number of prime factors of k.

Case 1: k=1. In this case, there is only on number, 1, that has order k.

Case 2 : k has a single prime factor. All the numbers with order k must be roots of the polynomial $x^k-1=0 \pmod{211}$. Since k is a prime, any number satisfying this polynominal has order either 1 or k. Sum of the roots of this polynomial (mod 211) is 0 and there is one of these numbers whose order is 1. Thus the sum of numbers of order k is $0-1=-1$.

Case 3 : k has two prime factors a, b. If $x^{ab}-1=1 \pmod{211}$, any number with order a or any number with order b satisfies this equation. Again the sum of all such numbers is 0 (mod 211). Hence the sum of numbers whose order is ab is

0 - (sum of numbers with order 1)

- (sum of numbers with order a)

- (sum of numbers with order b)

= 0-1-(-1)-(-1)

where we have used Case 1 to compute sum of numbers with order a and sum of numbers with order b. Thus in the

case, the sum of number of order ab can be written as $1=(-1)^2$.

Case 4 : k has three prime factors. Using the same logic as in Case 3, we get the sum of numbers of order k equals.

$$0-1-(-1)-(-1)-(-1)-(1)-(1)-(1) = -1 = (-1)^3$$

We see that, in general, the sum of numbers of order k is $(-1)^t$, where t is the number of prime factors k. Note that since 210 is square free, we need not worry about orders having squares of primes as factors. Thus our desired sum is

$$\sum_{K|210} k(-1)^{\text{the number of prime factors of k}}$$

The divisors of 210 are

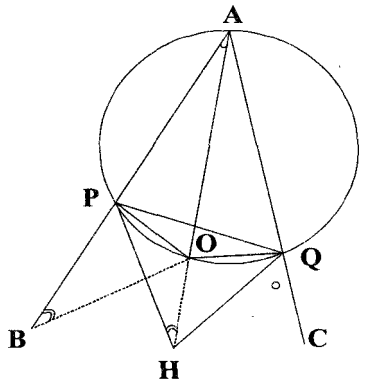
$$1,2,3,5,6,7,10,14,15,21,35,30,42,70,105,210$$

and thus the sum is

$$1-(2+3+5+7) +(6+10+14+15+21+35)$$
$$- (30+42+70+105) - 210 \pmod{211}$$
$$= (1-2)(1-3)(1-5)(1-7) \pmod{211} = 48$$

8. An acute angled triangle ABC is given. A circle passing through A and the circumcenter O of the triangle ABC intersects AB and AC at points P and Q respectively. Prove that the othocenter of the triangle POQ line on the line.

Solu : Denote the orthocenter by H. We will prove that the quadrilaterals BHOP and CHOQ are cyclic. This will give $\angle BHO+\angle OHC = \angle APO+\angle OQA=180°$. By symmetry, it suffices to prove that BHOC is cyclic.



$\angle OHP = 90^0 - \angle HPQ = \angle PQO = \angle PAO = \angle OBA$

Hence BHOP is cyclic and we are done.

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MATH LAB vs MATH PHOBIA

"Mathematics is king of all arts and queen of all sciences." This proverb is upto 20th century. In 21st century world evaluated as "math is nuclei of all subjects". To start with, UNESCO declared 2000 A.D as "year of mathematics".

Now-a-days, math is considered as neither arts subject nor science subjects. But science without math is meaningless. Even though of all importance math lab is not introduced in the school curriculum. The various inputs to learning of math have aimed at developing certain basic and essential skills for its use in daily life as well as in furtherance of knowledge. The complexity of the subject seems to have appeared in the content and the process involved in the learning of math especially at the school level. Concepts of math are seen as difficult to understand and appreciate. Math phobia increases the resistance of the learning process.

Motivating children to learn math with interest and involvement through appreciation of its intrinsic worth poses a challenge to practice teaching of mathematics and parents. The problem is to expose children to math in the environment and help them to engage themselves in manipulating ordinary objects to experiencing the mathematics inherent in such manipulations without resorting to a formal study of mathematics. Children will find themselves discovering math on their own, and that will develop in them a lifelong aptitude for math-imitative and repetitive

exercises continues to be considered respectable to teachers as it carries the backing of tradition.

Math lab and projects have become a part of school curriculum. Many math teachers find it difficult to implement teaching aids and projects and also parents are unable to guide their children who study math in various levels.

Abstract concepts in math arise from the analysis of concrete situations. To inculcate the students' power of understanding the abstract concepts, it is necessary to predict a result after observing an experiment repeatedly. This observation is not the result of math, but it must be proved by a logical argument which is known as 'proof of math.

"Math lab" concepts can be adopted to teach many of the results of the higher secondary school mathematics. Math teachers must observe the student to predict the result experimentally first and then to prove or disapprove the prediction by math methods. This "math lab" activity in math subject creates curiosity, questioning capability and learning interesting and pleasant. After the experiments, the students will inspired by sufficient motivation and interest among the students.

The need for adoption of strategies that provide children appropriate and adequate experiences for abstracting ideas and finding their way to get at concepts with long term benefits characterized by learning through self-confidence and initiative. The rate of drop-outs and continuing math phobia among the learners, remedial programmes cannot be successful without involvement in

concrete experiences. It is highly expected of a math teacher to arouse curiosity, to encourage questioning and thereby make learning interesting and pleasant. Face for the learning of the math subject increases the resistance to the learning process. Math lab will facilitate in doing simple experiment that would lead to empowerment of many skills among learners. They are Numerical skill observation skill, thinking skill, analytical skill, understanding logic, skills of comparing, skills of interpretation, problem solving skills, decision making skills, special analysis and interpretation, life skills, skills of games, all skills could easily be developed by using indigenous materials like paper, sticks, pins,... etc. The objective of the math lab is to promote critical thinking and ability to approach a problem in different ways. This inculcates the imagination of the learners so that they are creative and contributive. It is essential that our students are exposes to think differently and approach problems without any fear. This activities is intended to motivate and sustain the interest level of the students and facilitate them to link their academic performance with other related activities.

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